Module - 1

Vector differenation

Important result :-

1. The position Vector defined on xy plane, yz plane, xz plane is $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and its d can be defined as $d\vec{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$ in other words if x, y, z are in function of t, then the position Vector $\vec{r} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ and its derivative is
$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

2. for any $F_1 = F_1(x,y,z)$, $F_2 = F_2(x,y,z)$, $\overline{F_3 = F_3(x,y,z)}$ Then the fun $\vec{F} = \overline{F_1 i} + \overline{F_2 j} + \overline{F_3 k}$ is called The Vector point

3. Suppose the two Vectors point fun $\vec{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\vec{B} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$. Then $\bar{A}.\bar{B} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \cdot (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$
$a_1b_1 + a_2b_2 + a_3b_3$ is a Scalar

(ii) $$\vec{A} \times \vec{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= (a_2b_3 - a_3b_2)\bar{i} - (b_3a_1 - a_3b_1)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$ is a Vector point function

4. The gradient of a Vector can be denoted by The Symbol ∇ read as del or grad and which will be denoted as
$$\nabla = \frac{\partial \bar{i}}{\partial x} + \frac{\partial \bar{j}}{\partial y} + \frac{\partial \bar{k}}{\partial z}$$
Suppose $\phi(x,y,z) = c$ be a Scalar point function. then The gradiant of ϕ can be defined as
$$\nabla\phi = \cancel{\partial\phi}\ \text{grad}\phi = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial \bar{k}}{\partial z}\right)\phi$$

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}}$$

5. Suppose $\phi_1(x,y,z) = c_1$, $\phi_2(x,y,z) = c_2$ are the two Scalar functions then

(i) $\nabla(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1\nabla\phi_1 + \lambda_2\nabla\phi_2$

(ii) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(iii) The angle between the given two Surfaces can be Evaluated by using $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

6. Suppose $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a Vector point function for which F_1, F_2, F_3 are the functions of x, y, z then

(i) $\operatorname{div}\vec{F}$ = can be defined as

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot\left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

$\operatorname{div}\vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ is a Scalar point function

(ii) $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial y}, \frac{\partial F_2}{\partial z}\right)\bar{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

is a Vector point function

7. If $\operatorname{div}\vec{F} = 0$ then we Say that $\vec{F}$ is a Solenoidal and if $\vec{F} = \bar{0}\bar{i} + \bar{0}\bar{j} + \bar{0}\bar{k} = \bar{0}$ then we Say that the given $\vec{F}$ is a Irrotational at any point $P(x_0, y_0, z_0)$

1) Find the angle between the surface $x^2+y^2+z^2=9$, $x^2+y^2-3=z$ at the point $(2,-1,2)$

$\Rightarrow$

$$\phi_1 = x^2+y^2+z^2-9=0$$
$$\phi_2 = x^2+y^2-3-z=0$$

$$\nabla\phi_1 = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$
$$\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$
$$\nabla\phi_1 P(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} + 2(2)\bar{k}$$
$$\Rightarrow \nabla\phi_1 P(2,-1,2) = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$\phi_2 = x^2+y^2-3-z=0$$
$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$$
$$\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - 1\bar{k}$$
$$\nabla\phi_2 P(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} - 1\bar{k}$$
$$\Rightarrow \nabla\phi_2 P(2,-1,2) = 4\bar{i} - 2\bar{j} - \bar{k}$$

$$\nabla\phi_1 = 4\bar{i} - 2\bar{j} + 4\bar{k} = \bar{A}$$
$$\nabla\phi_2 = 4\bar{i} - 2\bar{j} - \bar{k} = \bar{B}$$

$$\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|\bar{A}||\bar{B}|}$$

$$\cos\theta = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{(4)^2+(-2)^2+(4)^2}\sqrt{(4)^2+(-2)^2+(-1)^2}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

2] Find the angle between the Surfaces $x^2+y^2-z^2=4$ and $x^2+y^2-13=z$ at The point $(2,1,2)$

$\Rightarrow$

$$\phi_1 = x^2+y^2-z^2-4=0$$

$$\phi_2 = x^2+y^2-13-z=0$$

$$\nabla\phi_1 = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} - 2z\bar{k}$$

$$\nabla\phi_1 p(2,1,2) = 2(2)\bar{i} + 2(1)\bar{j} - 2(2)\bar{k}$$

$$\Rightarrow \nabla\phi p(2,1,2) = 4\bar{i} + 2\bar{j} - 4\bar{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$$

$$\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - 1\bar{k}$$

$$\nabla\phi_2 p(2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} - 1\bar{k}$$

$$\Rightarrow \nabla\phi_2 p(2,-1,2) = 4\bar{i} - 2\bar{j} - \bar{k}$$

$$\nabla\phi_1 = 4\bar{i} + 2\bar{j} - 4\bar{k} \Rightarrow \bar{A}$$

$$\nabla\phi_2 = 4\bar{i} + 2\bar{j} - 1\bar{k} \Rightarrow \bar{B}$$

$$\cos\theta = \frac{\bar{A}\cdot\bar{B}}{|\bar{A}||\bar{B}|}$$

$$\cos\theta = \frac{(4\bar{i}+2\bar{j}-4\bar{k})\cdot(4\bar{i}+2\bar{j}-1\bar{k})}{\sqrt{(4)^2+(2)^2+(4)^2}\sqrt{(4)^2+(2)^2+(-1)^2}}$$

$$\cos\theta = \frac{16+4+4}{6\sqrt{21}}$$

$$\cos\theta = \frac{24}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)}$$

3] Find the Values a and b Such that Surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1,-1,2)$

$\Rightarrow$

$$\phi_1 = ax^2 - byz - (a+2)x = 0 \rightarrow ①$$

$$\phi_2 = 4x^2y + z^3 - 4 = 0 \rightarrow ②$$

the point $p(1,-1,2)$ is on both the Surfaces

$$① \Rightarrow a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$\Rightarrow a + 2b = a + 2$$

$$\therefore b = 1$$

$$① \Rightarrow \phi_1 = ax^2 - yz - (a+2)x = 0$$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_1}{\partial z}\bar{k}$$

$$\nabla\phi_1 = (2ax - a - 2)\bar{i} - z\bar{j} - y\bar{k}$$

$$\nabla\phi\, p(1,-1,2) = 2a(1) - a - 2)\bar{i} - 2\bar{j} + \bar{k}$$

$$\nabla\phi_1 = (a-2)\bar{i} - 2\bar{j} + \bar{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\bar{i} + \frac{\partial\phi_2}{\partial y}\bar{j} + \frac{\partial\phi_2}{\partial z}\bar{k}$$

$$\nabla\phi_2 = 8xy\bar{i} + 4x^2\bar{j} + 3z^3\bar{k}$$

$$\nabla\phi_2\, p(1,-1,2) = -8(\bar{i}) + 4\bar{j} + 12\bar{k}$$

Given that ϕ_1 and ϕ_2 Orthogonally Intersect

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$|(a-2)\bar{i} - 2\bar{j} + \bar{k}| \cdot |-8\bar{i} + 4\bar{j} + 12\bar{k}| = 0$$

$$\Rightarrow -8(a-2) - 8 + 12 = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 = 0$$

$$\Rightarrow 20 - 8a = 0$$

$$\Rightarrow 20 = 8a$$

$$\Rightarrow a = \frac{20}{8}$$

$$\Rightarrow \boxed{a = \frac{5}{2}}$$

5 Find the angle between the Normals to the Surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

$\Longrightarrow$

$$xy = z^2$$

$$xy - z^2 = 0$$

$$\phi = xy - z^2$$

$$p = (4,1,2) \qquad q = (3,3,-3)$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$(\nabla\phi)_p = 1\bar{i} + 4\bar{j} - 4\bar{k} = \bar{A}$$

$$(\nabla\phi)_\phi = 3\bar{i} + 3\bar{j} + 6\bar{k} = \bar{B}$$

W.K.T $\quad \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

$$\cos\theta = \frac{(\bar{i} + 4\bar{j} - 4\bar{k})\cdot(3\bar{i} + 3\bar{j} + 6\bar{k})}{\sqrt{(1)^2 + (4)^2 + (4)^2}\ \sqrt{(3)^2 + (3)^2 + (6)^2}}$$

$$\cos\theta = \frac{3 + 12 - 24}{\sqrt{33}\ \sqrt{54}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\ 3\sqrt{6}}$$

$$\cos\theta = \frac{-9}{3\sqrt{33 \times 6}}$$

$$\cos\theta = \frac{-3}{\sqrt{198}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{198}}\right)}$$

6 If $\vec{F} = (xy^3z^2)$ (or) $\vec{F} = \text{grad}(xy^3z^2)$ find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$ at the point $(1,-1,1)$

$\Rightarrow$

$$\vec{F} = \nabla(xy^3z^2)$$

$$\vec{F} = \nabla\phi$$

$$\therefore \phi = xy^3z^2$$

$$\frac{\partial\phi}{\partial x} = y^3z^2, \quad \frac{\partial\phi}{\partial y} = 3y^2xz^2, \quad \frac{\partial\phi}{\partial z} = 2xy^3z$$

$$\vec{F} = \nabla\phi$$

$$\vec{F} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\vec{F} = y^3z^2\bar{i} + 3y^2xz^2\bar{j} + 2xy^3z\bar{k}$$

(i) $\text{div}\vec{F} = \nabla\cdot\vec{F}$

$$\text{div}\vec{F} = \left|\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right| \cdot \left|y^3z^2\bar{i} + 3y^2xz^2\bar{j} + 2xy^3z\bar{k}\right|$$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial(3y^2xz^2)}{\partial y} + \frac{\partial(2xy^3z)}{\partial z}$$

$$\text{div}\vec{F} = 0 + 6xyz^2 + 2xy^3$$

$$\text{div}\vec{F} = 6xyz^2 + 2xy^3$$

$$\text{div}\vec{F}_{P(1,-1,1)} = 6(1)(-1)(1)^2 + 2(1)(-1)^3$$

$$\text{div}\vec{F} = -6-2$$

$$\text{div}\vec{F} = -8$$

(ii) $\text{curl}\vec{F} = \nabla\times\vec{F}$

$$\text{curl}\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$\text{curl}\,\vec{F} = (6xy^2z - 6xy^2z)\bar{i} - (2y^3z - y^3 2z)\bar{j} + (3y^2z^2 - 3y^2z^2)\bar{k}$$

$$\text{curl}\,\vec{F} = \left[\left(6(1)(-1)^2(1) - 6(1)(-1)^2(1)\right)\bar{i}\right] - \left[2(-1)^3(1) - (-1)^3 2(1)\right]\bar{j}$$

$$+ \left[3(-1)^2(1) - 3(-1)^2(1)\right]$$

$$\text{curl}\,\vec{F} = [6-6]\bar{i} - [2-2]\bar{j} + [3-3]\bar{k}$$

$$\text{curl}\,\vec{F} = 0\bar{i} - 0\bar{j} + 0\bar{k}$$

$$\text{curl}\,\vec{F} = \vec{0} \quad \text{ie irrotational}$$

7) If $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$ find $\text{div}\,\vec{F}$ and $\text{curl}\,\vec{F}$

$\Longrightarrow$

$$\vec{F} = \nabla(x^3+y^3+z^3-3xyz) \longrightarrow (1)$$

$$\phi = x^3+y^3+z^3-3xyz$$

$$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz \Longrightarrow 3(x^2-yz)$$

$$\frac{\partial\phi}{\partial y} = 3y^2 - 3xz \Longrightarrow 3(y^2-xz)$$

$$\frac{\partial\phi}{\partial z} = 3z^2 - 3xy \Longrightarrow 3(z^2-xy)$$

$$(1) \Longrightarrow \vec{F} = \nabla\phi$$

$$\vec{F} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\vec{F} = 3(x^2-yz)\bar{i} + 3(y^2-xz)\bar{j} + 3(z^2-xy)\bar{k}$$

(i) $\text{div}\,\vec{F} = \nabla\cdot\vec{F}$

$$\text{div}\,\vec{F} = \left|\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right| \cdot \left|3(x^2-yz)\bar{i} + 3(y^2-xz)\bar{j} + 3(z^2-xy)\right|$$

$$\text{div}\,\vec{F} = 3\frac{\partial}{\partial x}(x^2-yz) + 3\frac{\partial}{\partial y}(y^2-xz) + 3\frac{\partial}{\partial z}(z^2-xy)$$

$$\text{div}\,\vec{F} = 6x + 6y + 6z$$

$$\text{div}\,\vec{F} = 6(x+y+z)$$

(ii) $curl\,\vec{F} = \nabla \times \vec{F}$

$$curl\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy) \end{vmatrix}$$

$$curl\,\vec{F} = (-3x+3x)\bar{i} - (-3y+3y)\bar{j} + (-3z+3z)\bar{k}$$

$$curl\,\vec{F} = \vec{0}$$

8] Find 'a' for which $\vec{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal

$\Rightarrow$ $\vec{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$

$\vec{F}$ is a solenoidal

$\Rightarrow div\,\vec{F} = 0$

$\Rightarrow \nabla \cdot \vec{F} = 0$

$\Rightarrow \left[\frac{\partial \bar{i}}{\partial x} + \frac{\partial \bar{j}}{\partial y} + \frac{\partial \bar{k}}{\partial z}\right] \cdot \left[(x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}\right] = 0$

$\Rightarrow 1 + 1 + a = 0$

$a + 2 = 0$

$\boxed{a = -2}$

9] Find the constants a, b and c such that $\vec{F} = (axy+bz^3)\bar{i} + (3x^2-cz)\bar{j} + (3xz^2-y)\bar{k}$ is irrotational also find the scalar potential such that $\vec{F} = \nabla\phi$

$\Rightarrow$ $\vec{F} = (axy+bz^3)\bar{i} + (3x^2-cz)\bar{j} + (3xz^2-y)\bar{k} \longrightarrow (1)$

Given that $curl\,\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \vec{0}$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+bz^3) & (3x^2-cz) & (3xz^2-y) \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$\Rightarrow (-1+0)\bar{i} - (3z^2 - 3bz^2)\bar{j} + (6x - ax)$

$\Rightarrow -1\bar{i} - 3z^2(1-b)\bar{j} + x(6-a) = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$3z^2(1-b) = 0 \qquad x(6-a) = 0 \qquad -1+c = 0$

$1-b = 0 \qquad 6-a = 0 \qquad c = 1$

$b = 1 \qquad 6 = a$

Scalar potential Equation ①

$\vec{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$

$\vec{F} = \nabla\phi$

$(6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 6xy + z^3$

$\Rightarrow \int \partial\phi = \int (6xy + z^3)\, dx$

$\phi = 3x^2y + xz^3 + f_1(y.z) \rightarrow ②$

$\frac{\partial\phi}{\partial y} = 3x^2 - z$

$\Rightarrow \int \partial\phi = \int 3x^2 - z\, \partial y$

$\phi = 3x^2y - yz + f_2(x.z) \longrightarrow ③$

$\frac{\partial\phi}{\partial z} = 3xz^2 - y$

$\Rightarrow \int \partial\phi = \int 3xz^2 - y\, dz$

$\phi = xz^3 - yz + f_3(x.y) \rightarrow ④$

from ② ③ and ④

$$\boxed{\phi = 3x^2y - yz + xz^3 + c}$$

10 Show that $\vec{F} = \frac{x\bar{i} + y\bar{j}}{x^2+y^2}$ is Solenoidal and irrotational

$\Longrightarrow$ $$\vec{F} = \frac{x\bar{i} + y\bar{j}}{x^2+y^2}$$

$$\vec{F} = \frac{x\bar{i}}{x^2+y^2} + \frac{y\bar{j}}{x^2+y^2} + 0\bar{k}$$

(i) $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

$$= \left[\frac{\partial \bar{i}}{\partial x} + \frac{\partial \bar{j}}{\partial y} + \frac{\partial \bar{k}}{\partial z}\right] \cdot \left[\frac{x}{x^2+y^2}\bar{i} + \frac{y}{x^2+y^2}\bar{j} + 0\bar{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + 0$$

$$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$\operatorname{div}\vec{F} = 0$$

$\vec{F}$ is a Solenoidal

(ii) $\operatorname{curl}\vec{F} = \nabla \times \vec{F}$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= (0-0)\bar{i} - (0-0)\bar{j} + \left[\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right]\bar{k}$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$= \vec{0}$$

curl $\vec{F} = \vec{0}$ it is a irrotational

11 Show that the vector field $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is an irrotational

$\Longrightarrow$ $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

curl $\vec{F} = \nabla \times \vec{F}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

curl $\vec{F} = (-1+1)\bar{i} - (-1+1)\bar{j} + (-1+1)\bar{k}$

curl $\vec{F} = 0\bar{i} + (1-1)\bar{j} + 0\bar{k}$

curl $\vec{F} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

curl $\vec{F} = \vec{0}$

$\therefore$ $\vec{F}$ is a irrotational

12 Find the constant a, b and c such that $\vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k}$ is an irrotational also find the scalar potential ϕ for which $\vec{F} = \nabla\phi$

$\Longrightarrow$ $\vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k} \rightarrow (1)$

$\vec{F}$ is an irrotational

$\Rightarrow$ curl $\vec{F} = \vec{0}$

$\Rightarrow \nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z & (x+cy+2z \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$\Rightarrow (c+1)\bar{i} - (1-a)\bar{j} + (b-1)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow (c+1)\bar{i} + (a-1)\bar{j} + (b-1)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$c = -1 \quad a = 1 \quad b = 1$

$= (a.b.c) \Rightarrow (-1, 1.1)$

By rewriting ① we get

① $\Rightarrow \vec{F} = (x+y+z)\bar{i} + (x+2y-z)\bar{j} + (x-y+2z)\bar{k}$

Given that $\vec{F} = \nabla\phi$

$\Rightarrow (x+y+z)\bar{i} + (x+2y-z)\bar{j} + (x-y+2z)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\frac{\partial\phi}{\partial x} = x+y+z$

$\Rightarrow \int \partial\phi = \int x+y+z \ \partial x$

$\phi = \frac{x^2}{2} + xy + zx + f_1(y.z) \longrightarrow ②$

$\frac{\partial\phi}{\partial y} = x+2y-z$

$\Rightarrow \int \partial\phi = \int x+2y-z \ \partial y$

$\phi = xy + y^2 - zy + f_2(x.z) \longrightarrow ③$

$\frac{\partial\phi}{\partial z} = x-y+2z$

$\Rightarrow \int \partial\phi = \int x-y+2z \ \partial z$

$\phi = xy - yz + z^2 + f_3(x.y) \longrightarrow ④$

from ② ③ and ④

$$\boxed{\phi = \frac{x^2}{2} + y^2 + z^2 + xy - zy + zx + C}$$

13] Find the constant a.b.c so that the vector field $\overline{F} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is an irrotational, also find the scalar potential ϕ for which $\overline{F} = \nabla\phi$

$$\Rightarrow \overline{F} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k} \rightarrow ①$$

Given that

$\overline{F}$ is an irrotational

$\Rightarrow$ curl $\overline{F} = \bar{0}$

$\Rightarrow \nabla \times \overline{F} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$\Rightarrow (c+1)\bar{i} - (4-a)\bar{j} + (b+2)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$c+1=0$ $\quad$ $4-a=0$ $\quad$ $b+2=0$

$c=-1$ $\quad$ $a=4$ $\quad$ $b=-2$

$\Rightarrow$ ① rewrite as

$$\overline{F} = (x+2y+4z)\bar{i} + (-2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$$

Given that $\overline{F} = \nabla\phi$

$$\Rightarrow (x+2y+4z)\bar{i} + (-2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$\frac{\partial\phi}{\partial x} = x - 2y + 4z$

$\Rightarrow \int \partial\phi = \int (x - 2y + 4z)\, dx$

$\phi = \frac{x^2}{2} - 2xy + 4xz + f_1(y, z) \longrightarrow (2)$

$\frac{\partial\phi}{\partial y} = -2x - 3y - z$

$\Rightarrow \int \partial\phi = \int (-2x - 3y - z)\, \partial y$

$\phi = -2yx - \frac{3y^2}{2} - zy + f_2(x, z) \longrightarrow (3)$

$\frac{\partial\phi}{\partial z} = 4x - y + 2z$

$\Rightarrow \int \partial\phi = \int (4x - y + 2z)\, \partial z$

$\phi = 4xz - yz + z^2 + f_3(x, y) \longrightarrow (4)$

from (2) (3) and (4)

$$\boxed{\vec{F} = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 - 2xy + 4xz - zy + C}$$

14\) Show that $\vec{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + (3xz^2)\bar{k}$ is irrotational (or) force field (or) conservative force (or) potential field. find the Scalar potential ϕ such that $\vec{F} = \nabla\phi$

$\Rightarrow \text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$= (0-0)\bar{i} - (3z^2 - 3z^2)\bar{j} + (2x - 2x)\bar{k}$

$= 0\bar{i} + 0\bar{j} + 0\bar{k}$

$= \vec{0}$ is a Irrotational

$\vec{F} = \nabla\phi$

$\Rightarrow (2xy + z^3)\bar{i} + x^2\bar{j} + (3xz^2)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\frac{\partial\phi}{\partial x} = 2xy + z^3$

$\Rightarrow \int \partial\phi = \int 2xy + z^3$

$\phi = x^2y + xz^3 + f_1(y.z) \longrightarrow (1)$

$\frac{\partial\phi}{\partial y} = x^2$

$\Rightarrow \int \partial\phi = \int x^2\, \partial y$

$\phi = x^2y + f_2(x,z) \longrightarrow (2)$

$\frac{\partial\phi}{\partial z} = 3xz^2$

$\Rightarrow \int \partial\phi = \int 3xz^2\, \partial z$

$\phi = xz^3 + f_3(x.y) \longrightarrow (3)$

from (1) (2) and (3)

$$\boxed{\phi = x^2y + xz^3 + c}$$

5 Find the constants a & b such that $\vec{F} = (axy+z^3)\bar{i} + (3x^2-z)\bar{j} + (bxz^2-y)\bar{k}$ is an irrotational and also find the scalar potential for which $\vec{F} = \nabla\phi$

$$\Rightarrow \quad \vec{F} = (axy+z^3)\bar{i} + (3x^2-z)\bar{j} + (bxz^2-y)\bar{k}$$

from given

$$\Rightarrow \operatorname{curl}\vec{F} = \vec{0}$$

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+z^3) & (3x^2-z) & (bxz^2-y) \end{vmatrix} = \overline{0i} + \overline{0j} + \overline{0k}$$

$$\Rightarrow (-1+1)\bar{i} - (bz^2 - 3z^2)\bar{j} + (6x - ax)\bar{k} = \overline{0i} + \overline{0j} + \overline{0k}$$

$$\Rightarrow \overline{0i} + (3-b)z^2\bar{j} + (6-a)x\bar{k} = \overline{0i} + \overline{0j} + \overline{0k}$$

$(3-b)z^2 = 0 \qquad (6-a)x = 0$

$3-b=0 \qquad 6-a=0$

$3=b \qquad a=6$

Rewrite equation ①

$$\vec{F} = (6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xy^2-y)\bar{k}$$

$$\vec{F} = \nabla\phi$$

$$(6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xy^2-y)\bar{k} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3$$

$$\Rightarrow \int \partial\phi = \int 6xy + z^3 \, \partial x$$

$$\phi = 3x^2y + xz^3 + f_1(y,z) \longrightarrow ②$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z$$

$$\Rightarrow \int \partial\phi = \int 3x^2 - z \, \partial y$$

$$\phi = 3x^2y - yz + f_2(x,z) \longrightarrow ③$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y$$

$$\Longrightarrow \int \partial\phi = \int 3xz^2 - y \, \partial z$$

$$\phi = xz^3 - yz + f_3(x,y) \longrightarrow (4)$$

from (2) (3) and (4)

$$\boxed{\phi = 3x^2y - yz + xz^3 + C}$$

Directional derivatives

16 Find the directional derivatives of $\phi = 4xz^3 - 3x^2y^2z$ and $(2,-1,2)$ along $2\bar{i} - 3\bar{j} + 6\bar{k}$

$$\Longrightarrow \quad \phi = 4xz^3 - 3x^2y^2z$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = (4z^3 - 6xy^2z)\bar{i} + (-6x^2yz)\bar{j} + (12xz^3 - 3x^2y^2)$$

$$\nabla\phi_{p(2,-1,2)} = (32-24)\bar{i} + 48\bar{j} + (96-12)\bar{k}$$

$$\nabla\phi = 8\bar{i} + 48\bar{j} + 84\bar{k}$$

and given that $\vec{d} = 2\bar{i} - 3\bar{j} + 6\bar{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{\sqrt{4+9+36}}$$

$$\hat{n} = \frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{7}$$

$$DD = \nabla\phi \cdot \hat{n}$$

$$DD = (8\bar{i} + 48\bar{j} + 84\bar{k}) \cdot \left(\frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{7}\right)$$

$$DD = \frac{1}{7}\left[16-144+504\right]$$

$$DD = \frac{376}{7}$$

$$\boxed{DD = 53.7143}$$

17 Find the directional derivative of $\phi = x^2+y^2+2z^2$ at $(1,2,3)$ along the direction of line $\overrightarrow{PQ} = 4\bar{i}-2\bar{j}+\bar{k}$

$\Longrightarrow$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

$$\nabla\phi\ P(1,2,3) = 2\bar{i} + 4\bar{j} + 12\bar{k}$$

and given that $\vec{d} = 4\bar{i}-2\bar{j}+\bar{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{(4)^2+(-2)^2+(1)^2}}$$

$$\hat{n} = \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{16+4+1}}$$

$$\hat{n} = \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{12}}$$

$$D.D = \nabla\phi\cdot\hat{n}$$

$$= (2\bar{i}+4\bar{j}+12\bar{k})\cdot\left(\frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{12}}\right)$$

$$= \frac{1}{\sqrt{12}}(8-8+12)$$

$$\boxed{DD = \frac{12}{\sqrt{12}}}$$

18 Find the DD of $xy^3+\cancel{x}yz^3$ at $(2,-1,1)$ in the direction of the Vector $\bar{i}+2\bar{j}+2\bar{k}$

$\Longrightarrow$

$$\phi = xy^3 + yz^3$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$$\nabla\phi = (y^3)\bar{i} + (3xy^2+z^3)\bar{j} + (3yz^2)\bar{k}$$

$$\nabla\phi_P = (-1)^3\bar{i} + (6+1)\bar{j} + (-3)\bar{k}$$

$$\nabla\phi_P = 1\bar{i} + 7\bar{j} - 3\bar{k}$$

$$\vec{d} = \bar{i}+2\bar{j}+2\bar{k}$$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{(1)^2+(2)^2+(2)^2}}$$

$$\hat{n} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$$

$$DD = \nabla\phi\cdot\hat{n}$$

$$= (-\bar{i}+7\bar{j}-3\bar{k})\cdot\left(\frac{\bar{i}+2\bar{j}+2\bar{k}}{3}\right)$$

$$= \frac{1}{3}\left(-1+14-6\right)$$

$$\boxed{DD = \frac{7}{3}}$$

19 If $\vec{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ show that $\vec{F} \cdot \text{curl}\,\vec{F} = 0$

$\Longrightarrow$ $\vec{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$\text{curl}\,\vec{F} = (-1-0)\bar{i} - (-1-0)\bar{j} + (0-1)\bar{k}$$

$$\text{curl}\,\vec{F} = (-1)\bar{i} - (-1)\bar{j} - \bar{k}$$

$$\therefore \vec{F} \cdot \text{curl}\,\vec{F} = |(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}| \cdot (-\bar{i} + \bar{j} - \bar{k}|$$

$$= (-x+y+1) + (1+x+y)$$

$$\boxed{\vec{F} \cdot \text{curl}\,\vec{F} = \underline{0}}$$

Vector Integration

Line integral :- Any integral which is to be evaluated along the curve c is called a line integral. Suppose $\vec{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a vector point function in $(x.y.z)$ and $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$ a position vector, then the line integral of c can defined as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c [f_1 dx + f_2 dy + f_3 dz] \cdot [dx\bar{i} + dy\bar{j} + dz\bar{k}]$$

$$\boxed{\int_c \vec{F} \cdot d\vec{r} = \int_c (f_1 dx + f_2 dy + f_3 dz)}$$

19 If $\vec{F} = 3xy\bar{i} - y^2\bar{j}$ Evaluate $\int_c \vec{F}\cdot\vec{dr}$ where c is the Variable of $y = 2x^2$ from point (0,0) to (1,2)

$\Rightarrow$ $\vec{F} = 3xy\bar{i} - y^2\bar{j} \rightarrow (1)$

Let $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow d\vec{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \vec{F}\cdot\vec{dr} = (3xy\bar{i} - y^2\bar{j})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\Rightarrow \vec{F}\cdot\vec{dr} = 3xy\,dx - y^2 dy \rightarrow (2)$

and given $y = 2x^2$

diff w.r.t 'x'

$\Rightarrow \frac{dy}{dx} = 4x$

$\Rightarrow dy = 4x\,dx$

(2) $\Rightarrow \vec{F}\cdot\vec{dr} = 3x(2x^2)dx - (2x^2)^2 4x\,dx$

$= 6x^3 dx - 16x^5 dx$

$\Rightarrow \vec{F}\cdot\vec{dr} = (6x^3 - 16x^5)dx$

$\therefore \int_c \vec{F}\cdot\vec{dr} = \int_c (6x^3 - 16x^5)\,dx$

$= \int_{x=0}^{1} (6x^3 - 16x^5)\,dx$

$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$

$= \frac{6}{4} - \frac{16}{6}$

$= \frac{3}{2} - \frac{8}{3}$

$= \frac{9-16}{6}$

$\boxed{\int_c \vec{F}\cdot\vec{dr} = \frac{-7}{6}}$

2) If $\vec{F} = x^2\bar{i} + xy\bar{j}$ Evaluate $\int \vec{F}\cdot\vec{dr}$ when from (0,0) to (1,1) along with the curves i) $y = x$

ii) $y = \sqrt{x}$

$\Rightarrow \vec{F} = x^2\bar{i} + xy\bar{j} \longrightarrow (1)$

Let $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow \vec{dr} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\therefore \vec{F}\cdot\vec{dr} = (x^2\bar{i} + xy\bar{j})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\Rightarrow \vec{F}\cdot\vec{dr} = x^2dx + xydy \longrightarrow (2)$

and given i) Towards the curves $c \Rightarrow y = x$

$\frac{dy}{dx} = 1$

$dy = dx$

$(2) \Rightarrow \vec{F}\cdot\vec{dr} = x^2dx + x.xdx$

$\vec{F}\cdot\vec{dr} = 2x^2dx$

$\int_c \vec{F}\cdot\vec{dr} = \int_c (2x^2)\,dx$

$= \int_0^1 (2x^2)\,dx$

$= \frac{2x^3}{3}\Big|_0^1 \,dx$

$\int_c \vec{F}\cdot\vec{dr} = \frac{2}{3}$

ii) Towards the curve $y = \sqrt{x}$

$\Rightarrow y^2 = x$

$2y = \frac{dx}{dy}$

$\Rightarrow dx = 2y\,dy$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$x = 2t\ ,\quad y = t\ ,\quad z = 3t$$

$$dx = 2t\ ,\quad dy = dt\ ,\quad dz = 3dt$$

$$① \Rightarrow \vec{F}\cdot\vec{dr} = 3(2t)^2\, 2\,dt + 2(3t)(2t) - t\,dt + 3t \cdot 3dt$$

$$= 24t^2\,dt + 12t^2\,dt - t\,dt + 3t\,9t$$

$$= (36t^2 + 8t)\,dt$$

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 (36t^2 + 8t)\,dt$$

$$= \left[\frac{36t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= \frac{36}{6} + \frac{8}{2}$$

$$= 12 + 4$$

$$\int_C \vec{F}\cdot\vec{dr} = 16$$

(ii) $$\vec{F}\cdot\vec{dr} = 3x^2\,dx + (2zx - y)\,dy + z\,dz \longrightarrow ①$$

$$x^2 - 4y \to ②$$

$$3x^2 - 8z \longrightarrow ③$$

$$x = 0,\quad x = 2$$

Let $x = t$

$$dx = dt$$

$$② \Rightarrow y = \frac{x^2}{4}$$

$$y = \frac{t^2}{4}$$

$$dy = \frac{2t}{4}\,dt$$

$$dy = \frac{1}{2}t\,dt$$

(2) $\Rightarrow \vec{F}\cdot\vec{dr} = (y^2)^2 2y\,dy + y(y^2)\,dy$

$$= 2y^5 dy + y^3 dy$$

$$\vec{F}\cdot\vec{dr} = dy(2y^5 + y^3)$$

$$\int_c \vec{F}\cdot\vec{dr} = \int_c (2y^5 + y^3)\,dy$$

$$= \int_0^1 (2y^5 + y^3)\,dy$$

$$= \left[\frac{2y^6}{6} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{2}{6} + \frac{1}{4}$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$\boxed{\int_c \vec{F}\cdot\vec{dr} = \frac{7}{12}}$$

Q1] Find the work done is moving a particle the force
given $\vec{F} = 3x^2\bar{i} + (2zx - y)\bar{j} + z\bar{k}$ along
i) A straight line from (0,0,0) to (2,1,3)
ii) The curve defined by $x^2 = 4y$, $3x^2 = 8z$ from $x=0$ to $x=2$

$\Rightarrow$

$$\vec{F} = 3x^2\bar{i} + (2zx - y)\bar{j} + z\bar{k}$$

Let $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\Rightarrow \vec{dr} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\vec{F}\cdot\vec{dr} = [3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}]\cdot[dx\bar{i} + dy\bar{j} + dz\bar{k}]$$

$$\vec{F}\cdot\vec{dr} = 3x^2 dx + (2zx - y)dy + z\,dz \longrightarrow (1)$$

i] Given the curve along the straight line passing through (0.0.0) to (1,2,3)

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

$x = 2t$, $y = t$, $z = 3t$

$dx = 2t$, dy, dt , $dz = 3dt$

① $\Rightarrow \vec{F}\cdot\vec{dr} = 3(2t)^2 2dt + 2(3t)(2t) - t\,dt + 3t \cdot 3dt$

$= 24t^2 dt + 12t^2 dt - t\,dt + 3t\,9t$

$= (36t^2 + 8t)\,dt$

$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_0^1 (36t^2 + 8t)\,dt$

$= \left[\frac{36t^3}{3} + 8\frac{t^2}{2}\right]_0^1$

$= \frac{36}{6} + \frac{8}{2}$

$= 12 + 4$

$\int_C \vec{F}\cdot\vec{dr} = 16$

(ii) $\vec{F}\cdot\vec{dr} = 3x^2 dx + (2zx - y)dy + z\,dz \longrightarrow$ ①

$x^2 = 4y \rightarrow$ ②

$3x^2 = 8z \longrightarrow$ ③

$x = 0$, $x = 2$

Let $x = t$

$dx = dt$

② $\Rightarrow$ $y = \frac{x^2}{4}$

$y = \frac{t^2}{4}$

$dy = \frac{2t}{4} dt$

$dy = \frac{1}{2} t\,dt$

$③ \Rightarrow \quad z = \frac{3x^2}{8}$

$z = \frac{3t^2}{8}$

$dz = \frac{9t^2}{8} dt$

$① \Rightarrow \vec{F}\cdot\vec{dr} = (3t^2)(dt) - \left[2\times\frac{3t^2}{8}\times t - \frac{t^2}{4}\right]\left[\frac{t^2}{2}dt\right] + \frac{3t^3}{8}\times\frac{9t^2}{8}dt$

$= \left[3t^2 + \left(\frac{3t^4}{4} - \frac{t^2}{4}\right)\frac{t}{2} + \frac{27t^5}{64}\right]dt$

$= \left[3t^2 + \frac{3t^5}{8} - \frac{t^3}{8} + \frac{27t^5}{64}\right]dt$

$= \left[3t^2 - \frac{t^3}{8} + \frac{51t^5}{64}\right]dt$

$\int_c \vec{F}\cdot\vec{dr} = \int_0^2 \left[3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right]dt$

$= \left[t^3 - \frac{t^4}{32} + \frac{51}{64}\frac{t^6}{6}\right]_0^2$

$= 8 - \frac{16}{32} + \frac{51}{384}(64)$

$= 8 - \frac{1}{2} + \frac{51}{6}$

$= \frac{48 - 3 + 51}{6}$

$= \frac{96}{6}$

$$\boxed{\int_c \vec{F}\cdot\vec{dr} = 16}$$

22] If $\vec{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ Evaluate $\int_C \vec{F}\cdot\vec{dr}$ from (0,0,0) to (1,1,1) along the curve given by $x=t$, $y=t^2$, $z=t^3$

$\Longrightarrow$ $\vec{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$

$\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\vec{dr} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\vec{F}\cdot\vec{dr} = [(3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}]\cdot[dx\bar{i} + dy\bar{j} + dz\bar{k}]$

$\vec{F}\cdot\vec{dr} = (3x^2+6y)dx - 14yz\,dy + 20xz^2dz$

$\vec{F}\cdot\vec{dr} = 3x^2 + 6y\,dx - 14yz\,dy + 20xz^2\,dz \rightarrow (1)$

$x = t \rightarrow (1)$ $\qquad z = t^3$

$(2) \Rightarrow dx = dt$ $\qquad (4) \Rightarrow dz = 3t^2dt$

$y = t^2 \rightarrow (2)$

$(3) \Rightarrow dy = 2t\,dt$

$(1) \Rightarrow \vec{F}\cdot\vec{dr} = (3t^2+6t^2)dt - 14t^2t^3\,2t\,dt + 20t\,t^6\,3t^2dt$

$\int_C \vec{F}\cdot\vec{dr} = \int_C (9t^2 - 28t^6 + 60t^9)\,dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= \left[3t^3 - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$

$= 3 - \frac{28}{7} + 6$

$= \frac{21 - 28 + 42}{7}$

$= \frac{35}{7}$

$\int_C \vec{F}\cdot\vec{dr} = 5$

Green's theorem :-

Statement :- If $M(x,y)$ & $N(x,y)$ be the two continous function of x & y having continous partial Integratives $\frac{\partial m}{\partial y}$ & $\frac{\partial n}{\partial x}$ in the region R of the xy - plane bounded by a closed curve

$$\int_{C} M(x,y)\,dx + N(x,y)\,dy = \iint_{R} \left(\frac{\partial m}{\partial x} - \frac{\partial n}{\partial y}\right) dx\,dy$$

$\boxed{23}$ Evaluate $\oint_{C} (3x^2 - 8y^2)dx + (4y - 6xy)$ where c is the boundary of the region Enclosed by $y = \sqrt{x}$ & $y = x^2$

$\Longrightarrow$

$$\oint_{C} M\,dx + \oint_{C} N\,dy = \oint_{C} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$\frac{\partial m}{\partial y} = -16y \qquad \frac{\partial n}{\partial x} = -6y$$

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = -6y + 16y$$

$$\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y} = 10y$$

The curve bounded by $y = \sqrt{x}$ and $y = x^2$

$\therefore$ By the Green's theoram w.r.t

$$\oint_{C} M\,dx + N\,dy = \iint_{R} \left(\frac{\partial m}{\partial x} - \frac{\partial n}{\partial y}\right) dx\,dy$$

$$= \iint_{R} (10y)\,dx\,dy$$

$$= 10 \int_{x=0}^{1} \int_{y=\sqrt{x}}^{x^2} y\,dy\,dx$$

$$= 10 \int_{0}^{1} \left[\frac{y^2}{2}\right]_{\sqrt{x}}^{x^2} dx$$

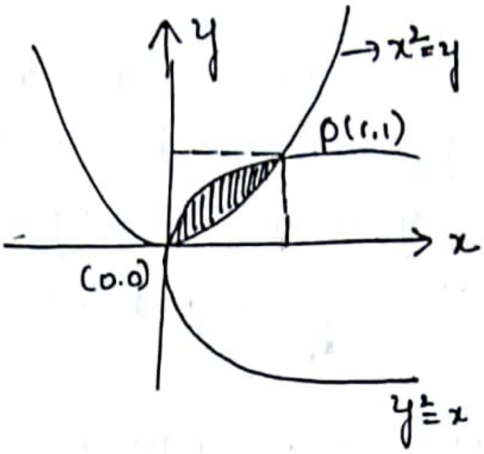
$$= 5\int_0^1 (x^4 - x)\,dx$$

$$= 5\left[\frac{x^5}{5} - \frac{x^2}{2}\right]_0^1$$

$$= 5\left[\frac{-3}{10}\right]$$

$$= \left|\frac{-3}{2}\right|$$

$$= \frac{3}{2}$$



34. Evaluate $\oint_c (xy+y^2)\,dx + x^2\,dy$, where c is the closed curve of the region bounded by $y=x$ and $y=x^2$ Using Green's theorem

$\Longrightarrow$ $\oint_c M\,dx + \oint_c N\,dy = \oint (xy+y^2)\,dx + x^2\,dy$

$M = xy+y^2 \qquad N = x^2$

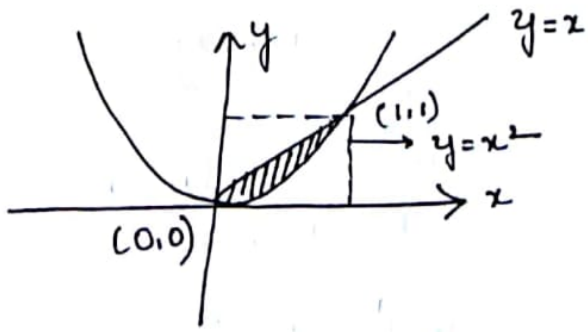
$\frac{\partial M}{\partial y} = x+2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - 2y$

The curve c bounded by $y=x$ & $y=x^2$

$\therefore$ By the Green's theorem. W.K.T



$$\oint_C M\,dx + N\,dy = \iint_{R_1}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow I = \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(x^2-x^2) - (x^3-x^4)\right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \frac{x^5}{5} - \frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$I = \frac{-1}{20}$$

25 Use Green's theorem to Evaluate $\oint (x^2+y^2)\,dx + 3x^2y\,dy$ where c is the circle $x^2+y^2=4$ traced in the positive Sigus Upper hay of the circle

$$\Rightarrow \oint_C M\,dx + \oint_C N\,dy = \oint (x^2+y^2)\,dx + 3x^2y\,dy$$

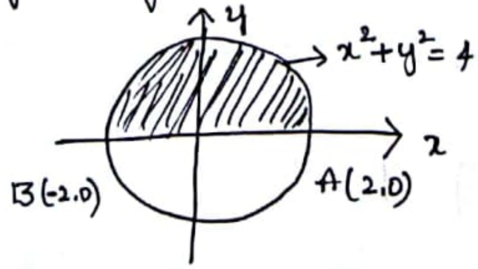
$M = x^2+y^2 \qquad N = 3x^2y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 6xy$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy - 2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2y(3x-1)$$

$\therefore$ By the green's theoram



$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} 24(3x-1)\,dy\,dx$$

$$= 2\int_{x=-2}^{2} (3x-1)(4-x^2)\,dx$$

$$= \int_{x=-2}^{2} (12y - 3x^2 - 4 + x^2)\,dx$$

$$= \int_{-2}^{2} (-3x^3 + x^2 + 12x - 4)\,dx$$

$$= \left[\frac{-3}{4}x^4 + \frac{x^3}{3} + 6x^2 - 4x\right]_{-2}^{2}$$

$$= \left[-12 + \frac{8}{3} + 24 - 8\right] - \left[-12 - \frac{8}{3} + 24 + 8\right]$$

$$= \cancel{-12} + \frac{8}{3} + \cancel{24} - 8 + \cancel{12} + \frac{8}{3} - \cancel{24} - 8$$

$$I = \frac{-32}{3}$$

$\boxed{26}$ Find the Area between the parabolas $y^2 = 4x$ & $x^2 = 4y$ by using Green's theorem

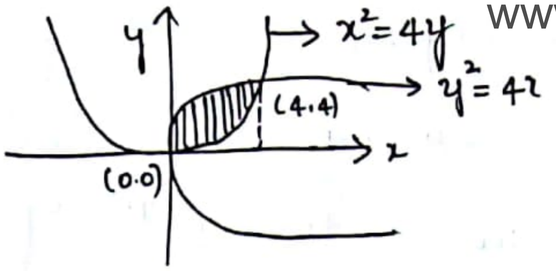
$\Longrightarrow$ $$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \longrightarrow (1)$$

$$\int_C M\,dx + N\,dy = \frac{1}{2}\int(-y\,dx + x\,dy)$$

$$M = \frac{-y}{2}, \quad N = \frac{x}{2}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2} \qquad \frac{\partial N}{\partial x} = \frac{1}{2}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} - \left(\frac{-1}{2}\right) = 1$$



$$① \Rightarrow \int M dx + N dy = \iint_{R_1} 1 \, dx \, dy$$

$$= \int_{x=0}^{4} \int_{y=x^2/4}^{2\sqrt{x}} (1) \, dy \, dx$$

$$= \int_{x=0}^{4} [y]_{x^2/4}^{2\sqrt{x}}$$

$$= \int_{x=0}^{4} \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= 2\int_0^4 x^{1/2} - \frac{1}{4}\int_0^4 x^2 dx$$

$$= \left[\frac{2x^{3/2}}{\frac{3}{2}}\right]_0^4 - \frac{1}{4}\left[\left(\frac{x^3}{3}\right)\right]_0^4$$

$$= \frac{4}{3}(4^{3/2}) - \frac{1}{12}(4^3)$$

$$= \frac{4\times 8}{3} - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3} \text{ sq. units}$$

27. Evaluate $\int_C (x^2+y^2)dx + 3x^2y\,dy$, where c is the circle $x^2+y^2=4$ traced in the positive sigus

$\Rightarrow$

$$\int M dx + N dy = \int_C (x^2+y^2)dx + 3x^2y\,dy$$

$M = x^2+y^2 \qquad N = 3x^2y$

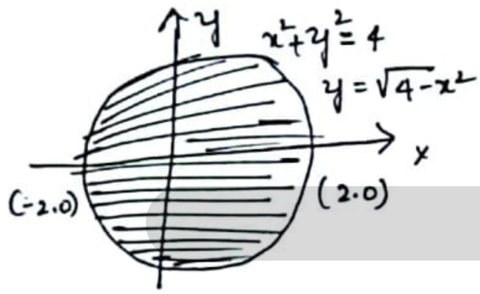
$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 6xy$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 6xy - 2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2y(3x-1)$$

And given that the circle Having the radius 2 and $x^2+y^2=4$ in the +ve Sence

$\therefore$ By green's theorem



$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_C (x^2+y^2)\,dx + 3x^2y\,dy = \int_{x=-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1)\,dy\,dx$$

$$= \int_{-2}^{2} 2(3x-1) \int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx$$

$$= \int_{-2}^{2} 2(3x-1) \left[\frac{y^2}{2}\right]_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} (3x-1)\left[(4-x^2)-(4-x^2)\right] dx$$

$$= 0$$

Stoke's theoram

Suppose S be an open surface bounded by a closed curve C & if $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be any vector point function having the continuous fun then $\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$, where $\hat{n}$ is the draw unit normal vector to the surface S (or)

$$\hat{n}\,ds = dy\,dz\,\bar{i} + dz\,dx\,\bar{i} + dx\,dy\,\bar{k}$$

$\boxed{28}$ Using Stoke's theoram for $\vec{F} = y\bar{i} + z\bar{j} + x\bar{k}$. where S is a upper half of the sphere $x^2+y^2+z^2=1$ and C is its boundary

$\Longrightarrow$ $\vec{F} = y\bar{i} + z\bar{j} + x\bar{k}$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= (0-1)\bar{i} - (1-0)\bar{j} + (0-1)\bar{k}$$

$$\text{Curl}\,\vec{F} = -\bar{i} - \bar{j} - \bar{k}$$

Given that surface 'S' is upper half of the sphere $x^2+y^2+z^2=1$ its Normal $\hat{n} = \bar{k}$

$\therefore$ $\text{curl}\,\vec{F} \cdot \hat{n} = (-\bar{i} - \bar{j} - \bar{k}) \cdot \bar{k} = -1$

By the Stoke's Theoram w.r.t

$$\oint \vec{F} \cdot \vec{dr} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

$$= \iint -1\, ds$$

$$= -\iint 1\, ds$$

$$= -A$$

where A is the Area of the Circle $x^2+y^2=1$ for which $z=0$

$$A = \pi r^2$$
$$= \pi \times 1$$
$$A = \pi$$
$$\therefore \int_c \vec{F} \cdot \vec{dr} = -\pi$$

Gauss divergence theoram :-

If $\vec{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is a Vector point function Having Continous function in the region V bounded by a Closed Surface S, then $\iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{N}\, ds$ where $\hat{n}$ is the Unit Normal Vector to the Surface S

$\boxed{29}$ Evaluate $\vec{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over the rectangular parallel piped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ find $\iiint_V \nabla \cdot \vec{F} \cdot dV$

$\Longrightarrow$ $\vec{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\therefore \operatorname{div}\vec{F} = \nabla \cdot \vec{F}$$

$$= \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right] \cdot \left[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy\bar{k})\right]$$

$$= 2x + 2y + 2z$$

$$\operatorname{div}\vec{F} = 2(x + y + z)$$

also given that given Surface is parallello piped bounded b/w $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

$$\iiint_V \operatorname{div}\vec{F} = \iiint_V 2(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x + y + z)\, dz\, dy\, dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xz+yz+\frac{z^2}{2}\right]_0^c dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xc+cy+\frac{c^2}{2}\right]dy\,dx$$

$$= 2\int_{x=0}^{a}\left[xy+cy^2+y\frac{c^2}{2}\right]_0^b dx$$

$$= 2\int_{0}^{a}\left[bc\frac{x^2}{2}+\frac{b^2cx}{2}+\frac{bc^2x}{2}\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= abc\,[a+b+c]$$

30. By using divergence theorem. Evaluate $\iint \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ & S is the surface enclosing the region for which $x^2+y^2 \le 4$, $0 \le z \le 3$,

$\Longrightarrow \quad \vec{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$

$div\,\vec{F} = \nabla\cdot\vec{F}$

$div\,\vec{F} = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\cdot\left(4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}\right)$

$\Longrightarrow \quad div\,\vec{F} = 4 - 4y + 2z$

z varies from 0 to 3

by using $x^2+y^2=4$

$\Longrightarrow y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

and x varies from -2 to 2

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\,\vec{F}\cdot dv$

$\Rightarrow I = \iiint_V (4 - 4y + 2z)\,dv$

$$I = 2\int_{x=-2}^{2}\int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(2-2y+z)\,dz\,dy\,dx$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[2z-2yz+\frac{z^2}{2}\right]\Big|_0^3\,dy\,dx$$

$$= 2\int_{x=-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(6-6y+9/2)\,dy\,dx$$

$$= 2\int_{x=-2}^{2}\left(6y-3y^2+\frac{9}{2}y\right)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx$$

$$= \int_{x=-2}^{2}\left[21y-6y^2\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\,dx$$

$$= \int_{x=-2}^{2}\left[21\sqrt{4-x^2}-6(4-x^2)\right]-\left[-21\sqrt{4-x^2}-6(4-x^2)\right]$$

$$= 42\int_{x=-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 42*2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 8\cancel{4}\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 8\cancel{4}*2*\frac{\pi}{4}$$

$$= \underline{84\pi}$$

31) $\vec{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$. Evaluate $\iiint_V \nabla\cdot\vec{F}\, dV$ where V is the region bounded by the planes $x=0, y=0, z=0$ $2x+2y+z=4$

$\Longrightarrow$ $\vec{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$

$\text{div}\vec{F} = \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right]\cdot\left[(2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}\right]$

$= 4x - 2x + 0$

$\text{div}\vec{F} = 2x$

The planes $x=0, y=0, z=0$ and $2x+2y+z=4$

$\therefore$ By the Gauss theorem we have

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\vec{F}\cdot dV$$

$$\Longrightarrow \quad I = \iiint_V 2x\, dV$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x\,[z]_0^{4-2x-2y}\, dy\, dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} (4x - 2x^2 - 2xy)\, dy\, dx$$

$$= 2\int_{x=0}^{2} \left(4xy - 2x^2y - xy^2\right)_0^{2-x} dx$$

$$= 2\int_{x=0}^{2} 4x(2-x) - 2x^2(2-x) - x^2(2-x)^2\, dx$$

$$= 2\int_{x=0}^{2} \left(8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^4 + 4x^3\right) dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\left[\frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2\right]_0^2$$

$$= 2\left[4 - \frac{32}{3} + 8\right]$$

$$= 2\left[12 - \frac{32}{3}\right]$$

$$= 2\left[\frac{4}{3}\right]$$

$$I = \frac{8}{3}$$

$\boxed{32}$ Use divergence Theoram to Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ Over the Entire Surface of the region above xy-plane bounded by the Cone $z^2 = x^2 + y^2$ and the plane $z = 4$,

$\vec{F} = 4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}$

$$\Longrightarrow \vec{F} = 4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}$$

$$\therefore \operatorname{div}\vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right) \cdot \left(4xz\bar{i} + xyz^2\bar{j} + 3z\bar{k}\right)$$

$$\operatorname{div}\vec{F} = 4z + xz^2 + 3$$

$x^2 + y^2 = z^2 \longrightarrow (1)$ The plane $z = 4$

$\therefore z$ Varies from 0 to 4

$$(1) \Rightarrow x^2 + y^2 = 4^2$$

$$x^2 + y^2 = 16 \longrightarrow (2)$$

When $y = 0$

$$(2) \Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

$$x = -1,\ x = 4$$

and $y^2 = 16 - x^2$

$y = \pm\sqrt{16-x^2}$

$y = +\sqrt{16-x^2}$ $\quad y = -\sqrt{16-x^2}$

W.K.T = $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\cdot dv$

$$\Rightarrow \iiint_V \operatorname{div}\vec{F}\cdot dv = \iiint_V (4z + xz^3 + 3)\,dv$$

$$= \int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{z=0}^{4} (4z + xz^2 + 3)\,dz\,dy\,dx$$

$$= \int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 2z^2 + 3\frac{z^3}{3} + 3z\Big|_0^4\,dy\,dx$$

$$= \int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(32 + \frac{64}{3}x + 12\right) dy\,dx$$

$$= \int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(44 + \frac{64x}{3} + \right) dy\,dx$$

$$= \frac{1}{3}\int_{x=-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (132 + 64x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x)\,dy\,dx$$

$$= \frac{4}{3}\int_{x=-4}^{4} (33+16x)[y]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \, dx$$

$$= \frac{4}{3}\int_{x=-4}^{4} (33+16x)\left(\sqrt{16-x^2}+\sqrt{16-x^2}\right) dx$$

$$= \frac{8}{3}\int_{x=-4}^{4} (16x+33)\sqrt{16-x^2}\, dx$$

$$= \frac{8}{3} * 16 \int_{x=-4}^{4} x\sqrt{16-x^2}\, dx + \frac{88}{3} \times 33 \int_{x=-4}^{4} \sqrt{16-x^2}\, dx$$

$$= \frac{128}{3}(0) + 88 \int_{x=-4}^{4} \sqrt{16-x^2}\, dx$$

$$= 176 \int_{0}^{4} \sqrt{16-x^2}\, dx$$

put $x = 4\sin\theta$

$dx = 4\cos\theta \, d\theta$

$\Longrightarrow \sin\theta = \frac{x}{4} \Longrightarrow \theta = \sin^{-1}(x/4)$

$x=4 \Rightarrow \theta = \pi/2$

$x=0 \Rightarrow \theta = 0$

$$I = 176 \int_{0}^{\pi/2} \sqrt{16-16\sin^2\theta} \, . \, (4\cos\theta) d\theta$$

$$= 176 \int_{0}^{\pi/2} 4^2\cos^2\theta \, d\theta$$

$$= 2816 \int_{0}^{\pi/2} \cos^2\theta \, d\theta$$

$$= 2816 * \frac{2-1}{2} * \pi/2$$

$$\boxed{I = 704\pi}$$

33] If $\vec{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$, where S is the Upper half Surface of the sphere $x^2+y^2+z^2=1$ is its boundary. Use Stokes theorem to find $\int_C \vec{F}\cdot\vec{dr}$

$\Rightarrow$ $\vec{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

$x^2+y^2+z^2=1$

$\therefore \text{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & -y^2z \end{vmatrix}$

$= (-2yz+2yz)\bar{i} - (0-0)\bar{j} + (0+1)\bar{k}$

$\text{curl}\vec{F} = \bar{k}$

W.K.T

$\hat{n}ds = dydz\bar{i} + dxdz\bar{j} + dxdy\bar{k}$

$\therefore \text{curl}\vec{F}\cdot\hat{n}ds = [\bar{k}]\cdot[dydz\bar{i} + dxdz\bar{j} + dxdy\bar{k}]$

$= dxdy$

Area of a circle $x^2+y^2=1$

$= \pi r^2$

$= \pi(1)$

$= \pi$

34] A Vector field is given by $\vec{F} = \sin y\bar{i} + x(1+\cos y)\bar{j}$ Evaluate the line integral over a Circular path given by $x^2+y^2=a^2$, $z=0$

$\Rightarrow$ $\vec{F} = \sin y\bar{i} + x(1+\cos y)\bar{j}$

$\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\vec{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\vec{F}\cdot\vec{dr} = [\sin y\,\bar{i} + x(1+\cos y)\bar{j} + 0\bar{k}]\cdot[dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}]$

$= \sin y\,dx + x(1+\cos y)dy$

$= \sin y\,dx + x\,dy + x\cos y\,dy$

$= [\sin y\,dx + x\cos y\,dy] + x\,dy$

$\Longrightarrow \vec{F}\cdot\vec{dr} = d[x\sin y] + x\,dy \longrightarrow (1)$

and given the curve $x^2+y^2=a^2 \longrightarrow (2)$

Let $x = a\cos\theta$ $\quad y = a\sin\theta$

$\Rightarrow dx = -a\sin\theta \quad dy = a\cos\theta$

θ varies form 0 to 2π

$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_C [x\sin y) + x\,dy]$

$= \int_0^{2\pi} a[a\cos\theta\,.\sin\theta(a\sin\theta)] + \int_0^{2\pi} a\cos\theta\,.\,a\cos\theta\,d\theta$

$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$

$= (0-0) + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$

$= a^2\int_0^{2\pi}\left[\frac{1+\cos 2\theta}{2}\right]_0 d\theta$

$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta$

$= \frac{a^2}{2}\left[0 + \frac{\sin 2\theta}{2}\right]_2^{2\pi}$

$= \frac{a^2}{2}\times 2\pi$

$\boxed{\int_C \vec{F}\cdot\vec{dr} = \pi a^2}$

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Introduction

Suppose $y = f(x)$ be a function in the variable of x then the Equation $a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots a_n y = Q$

$\Rightarrow a_0D^n y + a_1D^{n-1}y + a_2D^{n-2}y + \cdots a_n y = Q$ is called the Linear differential Equation of n^{th} order and 1^{st} degree where D is called differential operation is follows

1. If $a_0, a_1, a_2 \cdots a_n$ are the Constant and $Q = 0$ then Eq ① called the Linear Homogenous differential Equation of n^{th} order with Constant co-efficient
2. If $a_0, a_1, a_2 \cdots a_n$ are the Constant and $Q \neq 0$ then Equation ① is called the Linear Non-Homogenous differential Equation of the n^{th} order with Constant Co-Efficient
3. If $a_0, a_1, a_2 \cdots a_n$ are the fun^n in x and $Q = 0$ then Eq ① is called the Linear Homogenous differential Equation of n^{th} order with Variable Co-Efficient
4. If $a_0, a_1, a_2 \cdots a_n$ are the function in x and $Q \neq 0$ then Eq ① is called the Linear non-homogenous Differential equation of n^{th} order with Variable Co-Efficient

Solution of the Non-Homogenous D.E with Variable Co-Efficient

Step 1 :-

write the given differential Equation

$a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots a_n y = Q(x)$

(or) $f(D)y = Q(x)$

Step 2 :-

Evaluation of complimentary function

a) Identify $f(D)$ in the given d.E

b) Write the Auxilary Equation $f(m)=0$, where this Equation will be polynomially in m of degree n. and It gives n number of Solutions they are $m = m_1, m_2, m_3 ---- m_n$

c) If $m_1, m_2, m_3 ----- m_n$ are real and distinct, then the Soln is $CF = y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} --- + c_n e^{m_n x}$

If first 2 roots are Equal and rest of them are real and distinct, then

$$CF = y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} ---- + c_n e^{m_n x}$$

If first 3 roots are Equal and rest of them are real and distinct, then

$$CF = y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + ---- c_n e^{m_n x}$$

If first 2 roots are complex and rest of them real and distinct, then

$$m_1 \pm m_2, m_3, m_4 ------- m_n$$

$$CF = y_c = [c_1 \cos m_2 x + c_2 \sin m_2 x] e^{m_1 x} + c_3 e^{m_3 x} + ----- ----- + c_n e^{m_n x}$$

If first 4 complex roots are Equal and rest of them real and distinct, then

$$CF = [(c_1 + c_2 x) \cos m_2 x + (c_3 + c_4 x) \sin m_2 x] e^{m_1 x} + c_3 e^{m_3 x} --- ---------- + c_n e^{m_n x}$$

Evaluation of particular Integral

$\Longrightarrow$ write the given DE in the form of

$f(D)y = Q(x)$

$\Rightarrow$ The particular integral evaluated by writing

$$y_p = \frac{\phi(x)}{f(D)}$$

If $f(a)=0$. Then $D-a$ should be factor to the $f(D)$ which will be evaluated as

$$\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)^k \phi(D)} = \frac{x^k}{k!}\frac{e^{ax}}{\phi(a)}$$

If $\phi(x) = \cos ax$ (or) $\sin ax$, then y_p becomes $\frac{\cos ax}{f(D)}$ (or) $\frac{\sin ax}{f(D)}$ which can be evaluated by replacing $D^2 = -a^2$ when $f(a) \neq 0$ and we have $\frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax$.

$$\frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax$$

1) Solve $(D^3+6D^2+11D+6)y=0$ where $D=\frac{d}{dx}$

$\Rightarrow$ Given $(D^3+6D^2+11D+6)y=0$

$f(D)y=0$

$f(D) = D^3+6D^2+11D+6$

$\therefore$ The Auxillary Equation is :

$f(m)=0$

$m=-1$	1	6	11	6
	0	-1	-5	-6
	1	5	6	0

$\Rightarrow m^3+6m^2+11m+6=0$

$\Rightarrow (m+1)(m^2+5m+6)=0$

$\Rightarrow m+1=0 \quad m^2+2m+3m+6=0$

$m=-1, \quad m(m+2)+3(m+2)=0$

$m=-1, \quad m+2=0 \quad m+3=0$

$m=-1, \quad m=-2, \quad m=-3$

$m=-1, -2, -3$

$\therefore CF = y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

2] Solve $\left(4\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 23\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y\right) = 0$

$\Rightarrow (4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

$\Rightarrow f(D)y = 0$

where $f(D) = 4D^4 - 4D^3 - 23D^2 + 12D + 36$

the Auxillary Equation is $f(m) = 0$

$\Rightarrow 4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

$\Rightarrow (m-2)(4m^3 + 4m^2 - 15m - 18)$

m=2	4	-4	-23	12	26
	0	8	8	-30	-36
+2	4	4	-15	-18	0
	0	8	24	18	
	4	12	9	0	

$\Rightarrow (m-2)(m-2)(4m^2 + 12m + 9) = 0$

$\Rightarrow m = 2, 2, \; 4m^2 + 6m + 6m + 9 = 0$

$\Rightarrow m = 2, 2, \; 2m(2m+3) + 3(2m+3)$

$\Rightarrow m = 2, 2, \; (2m+3)(2m+3)$

$\Rightarrow m = 2, 2 \quad m = \frac{-3}{2}, \; m = \frac{-3}{2}$

$\therefore y_c = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-3/2x}$

3] Solve $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6\right)y = e^x + 1$

$\Rightarrow$ Given $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6\right)y = e^x + 1 \longrightarrow (1)$

$\Rightarrow (D^3 + 6D^2 + 11D + 6)y = e^x + 1$

$f(D)y = e^x + 1$

when $f(D) = D^3 + 6D^2 + 11D + 6$

-1	1	6	11	6
	0	-1	-5	-6
	1	5	6	0

the A.E $f(m) = 0$

$m^3 + 6m^2 + 11m + 6 = 0$

$\Rightarrow (m+1)(1m^2+5m+6)=0$

$\Rightarrow (m+1)(m+2)(m+3)=0$

$\Rightarrow m=-1,\ m=-2,\ m=-3$

$\therefore\ y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

To find PI

$$PI = y_p = \frac{e^x+1}{f(D)}$$

$$= \frac{e^x}{f(D)} + \frac{1}{f(D)}$$

$$= \frac{e^x}{D^3+6D^2+11D+6} + \frac{e^{0x}}{0+0+0+6}$$

$$= \frac{e^x}{24} + \frac{1}{6}$$

$$y_p = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-3x} + \frac{e^x}{24} + \frac{1}{6}$$

$\boxed{4}$ Solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

$\Rightarrow (D^2-4)y = \cosh(2x-1) + 3^x$

$f(D)y = \cosh(2x-1) + 3^x$

where $f(D) = D^2 - 4$

To find CF

the AE is $f(m)=0$

$\Rightarrow m^2-4=0$

$\Rightarrow m=\pm 2$

$\therefore\ y_c = c_1 e^{-2x} + c_2 e^{2x}$

To find PI

$$PI = y_p$$

$$y_p = \frac{\cosh(2x-1)}{f(D)} + \frac{3^x}{f(D)}$$

$$y_p = \frac{\frac{e^{(2x-1)} + e^{-(2x-1)}}{2}}{D^2-4} + \frac{e^{(\log_e 3)x}}{D^2-4}$$

$$y_p = \frac{1}{2} \cdot \frac{e^{(2x-1)} + e^{-(2x+1)}}{D^2-4} + \frac{e^{(\log 3)x}}{D^2-4}$$

$$y_p = \frac{1}{2} \cdot \frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{e^{2x-1}}{(D-2)'(D+2)} + \frac{1}{2} \frac{e^{-2x+1}}{(D+2)'(D-2)} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{x}{1!} \frac{e^{2x-1}}{2+2} + \frac{1}{2} \frac{e^{-2x+1}}{(-2-2)} + \frac{e^{\log 3x}}{(\log 3)^2-4}$$

$$y_p = \frac{x}{1!} \frac{e^{2x-1}}{2+2} + \frac{1}{2} \frac{e^{-2x+1}}{(-2-2)} + \frac{e^{\log 3x}}{(\log 3)^2-4}$$

$$y_p = \frac{x}{8} e^{2x-1} - \frac{x}{8} e^{-(2x-1)} + \frac{3^x}{(\log 3)^2-4}$$

$$y_p = \frac{x}{4}\left[\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right] + \frac{3^x}{(\log 3)^2-4}$$

$$y_p = \frac{x}{4} \sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} + \frac{x}{4} \sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}}$$

6

5 Solve $(D^2+2D+1)y = 2x+x^2$, where $D=\frac{d}{dx}$

$\Longrightarrow$ Given

$\Rightarrow (D^2+2D+1)y = 2x+x^2$

$f(D)y = 2x+x^2$

where $f(D) = D^2+2D+1 = (D+1)^2$

The A.E is $f(m)=0$

$\Rightarrow (m+1)^2=0$

$\Rightarrow m=-1, -1$

$\therefore y_c = (c_1+c_2x)e^{-x}$

$y_p = \frac{x^2+2x}{f(D)}$

$y_p = \frac{x^2+2x}{(D+1)^2}$

$y_p = (1+D)^{-2}(x^2+2x)$

$y_p = (1-2D+3D^2-4D^3+---)(x^2+2x)$

$y_p = (x^2+2x)-2(2x+2)+3(2)$

$y_p = x^2+2x-4x-4+6$

$y_p = x^2-2x+2$

$\therefore y = y_c+y_p$

$$\boxed{y = (c_1+c_2x)e^{-x}+x^2-2x+2}$$

6 Solve $(D^2-4D+4)y = 8(e^{2x}+\sin 2x)$, where $D=\frac{d}{dx}$

$\Longrightarrow$ Given

$(D^2-4D+4)y = 8$

$f(D)y=8$

where $f(D) = D^2-4D+4$

To find CF

the A.E is $f(m)=0$

$$\Rightarrow m^2-4m+4=0$$

$$\Rightarrow (m-2)^2=0$$

$$\Rightarrow m=+2,-2$$

$$\therefore y_c=(c_1+c_2x)e^{2x}$$

To find PI

$$y_p=\frac{8(e^{2x}+\sin 2x)}{f(D)}$$

$$y_p=\frac{8e^{2x}}{f(D)}+\frac{8\sin 2x}{f(D)}$$

$$y_p=\frac{8e^{2x}}{(D-2)^2}+\frac{8\sin 2x}{D^2-4D+4}$$

$$y_p=8\frac{x^2}{2!}e^{2x}+8.\frac{\sin 2x}{(-4-4D+4)}$$

$$y_p=4x^2e^{2x}-\frac{8}{4}\frac{\sin 2x}{D}$$

$$y_p=4x^2e^{2x}-2D\frac{\sin 2x}{D^2}$$

$$y_p=4x^2e^{2x}-2D\frac{\sin 2x}{(-4)}$$

$$y_p=4x^2e^{2x}+\frac{1}{2}(2\cos 2x)$$

$$y_p=4x^2e^{2x}+\cos 2x$$

$$y=y_c+y_p$$

$$y=(C_1+c_2x)e^{2x}+4x^2e^{2x}+\cos 2x$$

7] $(D^3+D^2-4D-4)y = 3e^{-x}-4x-6$ using Inverse differential Equation

$\Rightarrow$

Given :- $(D^3+D^2-4D-4)y = 3e^{-x}-4x-6$

$$f(D)y = 3e^{-x}-4x-6$$

when $f(D) = D^3+D^2-4D-4$

To find CF

The A.E is $f(m) = 0$

$$\Rightarrow m^3+m^2-4m-4=0$$

$$\Rightarrow (m+1)(m+2)(m-2)=0$$

$$\Rightarrow m=-1,\ m=-2,\ m=2$$

$$\therefore y_c = c_1e^{-x} + c_2e^{-2x} + c_3e^{2x}$$

To find PI

$$y_p = \frac{3e^{-x}-4x-6}{f(D)}$$

$$y_p = \frac{3e^{-x}}{D^3+D^2-4D-4} - \frac{6}{D^3+D^2-4D-4} - \frac{4x}{D^3+D^2-4D-4}$$

$$y_p = 3\frac{e^{-x}}{(D+1)(D^2-4)} - 6\frac{e^{0x}}{D^3+D^2-4D-4} - 4\cdot\frac{x}{(-4)\left[1 - \frac{(D)^3+(D)^2-4D}{4}\right.}$$

$$y_p = \frac{3x^1}{1!}\frac{e^{-x}}{[(-1)^2-4]} - 6\frac{e^{0x}}{(-4)} + \left[1 - \left(\frac{D^3+D^2-4D}{4}\right)\right]^{-1}x$$

$$y_p = -xe^{-x} + \frac{3}{2} + \left[1 + \left(\frac{D^3+D^2-4D}{4}\right) + \cdots\right]x$$

$$y_p = -xe^{-x} + \frac{3}{2} + x + \frac{1}{4}(-4)$$

$$y_p = -xe^{-x} + \frac{3}{2} + x - 1$$

$$y_p = -xe^{-x} + x + \frac{1}{2}$$

$$y = y_c + y_p$$

$$y = c_1e^{-x} + c_2e^{-x} - xe^{-x} + x + 1/2$$

8. Solve $(D^3+8)y = x^4+2x+1$ where $D = \frac{d}{dx}$

$\Rightarrow$ Given :- $(D^3+8)y = x^4+2x+1$

$\Rightarrow f(D)y = x^4+2x+1$

when $f(D) = D^3+8$

To find CF

The A.E is $f(m)=0$

$\Rightarrow m^3+8=0$

$\Rightarrow m^3+2^3=0$

$\Rightarrow (m+2)(m^2-2m+4)=0$

$\Rightarrow m+2=0,\ m^2-2m+4=0$

$m=-2,\quad m = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$m = \frac{2\pm\sqrt{4-16}}{2}$

$m = 2\pm\sqrt{-12}$

$m = \frac{2\pm 2\sqrt{3}i}{2}$

$m = 1\pm i\sqrt{3}$

$\therefore m = -2,\ 1\pm i\sqrt{3}$

$y = c_1e^{-2x} + [c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x]e^x$

To find PI

$y_p = \frac{x^4+2x+1}{f(D)}$

$y_p = \frac{x^4+2x+1}{D^3+8}$

$y_p = \frac{1}{8}\cdot\frac{x^4+2x+1}{1+\left(\frac{D^3}{8}\right)}$

$$y_p = \frac{1}{8}\left[1 - \frac{D^3}{8} + \frac{D^6}{64} - - - -\right](x^4+2x+1)$$

$$y_p = \frac{1}{8}\left[x^4+2x+1 - \frac{1}{8}(24x)\right]$$

$$y_p = \frac{1}{8}\left[x^4+2x+1 - 3x\right]$$

$$y_p = \frac{1}{8}(x^4 - x + 1)$$

$$\therefore\ y = y_c + y_p$$

$$y = c_1 e^{-2x} + \left[c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right]e^{x} + \frac{1}{8}(x^4 - x + 1)$$

9] Solve $(D^2+4)y = x^2 + e^{-x}$ using Inverse differential operation method

$\Longrightarrow$ Given :- $(D^2+4)y = x^2 + e^{-x}$

$\Rightarrow f(D)y = x^2 + e^{-x}$

when $f(D) = D^2+4$

To find CF

The A.E is $f(m) = 0$

$\Longrightarrow m^2+4 = 0$

$\Longrightarrow m^2 = -4$

$\Longrightarrow m = \pm 2i$

$\Longrightarrow m = 0 \pm 2i$

$\therefore\ y_c = c_1\cos 2x + c_2 \sin 2x$

To find PI

$$y_p = \frac{x^2 + e^{-x}}{D^2+4}$$

$$y_p = \frac{x^2}{D^2+4} + \frac{e^{-x}}{D^2+4}$$

$$y_p = \frac{x^2}{4\left(1+\frac{D^2}{4}\right)} + \frac{e^{-x}}{(-1)^2+4}$$

$$y_p = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2 + \frac{e^{-x}}{5}$$

$$y_p = \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}+ - - - - -\right]x^2 + \frac{e^{-x}}{5}$$

$$y_p = \frac{1}{4}\left[x^2 - \frac{1}{4}(2)\right] + \frac{e^{-x}}{5}$$

$$y_p = \frac{x^2}{4} + \frac{e^{-x}}{5} - \frac{1}{8}$$

$\therefore$ $y = y_c + y_p$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x^2}{4} + \frac{e^{-x}}{5} - \frac{1}{8}$$

$\boxed{10}$ Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 1 + 3x + x^2$

Given :- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 1 + 3x + x^2$

$\Rightarrow (D^2 + 3D + 2)y = x^2 + 3x + 1$

$\Rightarrow f(D)y = x^2 + 3x + 1$

when $f(D) = D^2 + 3D + 2$

The A.E is $f(m) = 0$

$\Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow m(m+1) + 2(m+1) = 0$

$\Rightarrow m+1 = 0,\ m+2 = 0$

$\Rightarrow m = -1,\ m = -2$

$\therefore$ $y = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{x^2+3x+1}{D^2+3D+2}$$

$$y_p = \frac{1}{2}\,\frac{x^2+3x+1}{\left[1+\frac{D^2+3D}{2}\right]}$$

$$y_p = \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}(x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}(D^2+3D)^2 - - - -\right](x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}[D^4+6D^3+9D^2) - - - -\right](x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[(x^2+3x+1) - \frac{1}{2}[2+3(2x+3)] + \frac{1}{4}[D+D+9(2)]\right]$$

$$y_p = \frac{1}{2}\left[x^3+3x+\frac{1}{2}(6x+11)+\frac{9}{2}\right]$$

$$y_p = \frac{1}{2}\left[x^3+3x-3x+1-\frac{11}{2}+\frac{9}{2}\right]$$

$$y_p = \frac{x^2}{2}$$

$$\therefore\quad y = y_c + y_p$$

$$y = c_1e^{-x}+c_2e^{-2x}+\frac{x^2}{2}$$

11. Solve $\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} + \frac{dy}{dx} = x^3$

$\Longrightarrow$ Given :- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3$

$\Longrightarrow$ $D^3+2D^2+Dy = x^3$

$\Longrightarrow$ $f(D)y = x^3$

when $f(D) = D^3+2D^2+D$

The A.E is $f(m)=0$

$$\Rightarrow m^3+2m^2+m=0$$

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m(m+1)^2=0$$

$$\Rightarrow m=0,\ m=-1,-1$$

$$\therefore y_c = c_1+(c_2+c_3x)e^{-x}$$

$$y_p = \frac{x^3}{f(D)}$$

$$y_p = \frac{x^3}{D^3+2D^2+D}$$

$$y_p = \frac{x^3}{D(D^2+2D+1)}$$

$$y_p = \frac{\frac{1}{D}x^3}{(D+1)^2}$$

$$y_p = \frac{\int x^3\,dx}{(1+D)^2}$$

$$y_p = \frac{1}{4}\frac{x^4}{(1+D)^2}$$

$$y_p = \frac{1}{4}(1+D)^{-2}x^4$$

$$y_p = \frac{1}{4}\left[1-2D+3D^2-4D^3+5D^4 - - - -\right]x^4$$

$$y_p = \frac{1}{4}\left[x^4-8x^3+3(12x^2)-4(24x)+5(24)\right]$$

$$y_p = \frac{1}{4}\left[x^4-8x^3+36x^2-96x+120\right]$$

$$\therefore y = y_c+y_p$$

$$y = c_1+(c_2+c_3x)e^{-x}+\frac{1}{4}\left[x^4-8x^3+36x^2-96x+120\right]$$

12. Solve $(D^2+4)y = x^2+\cos 2x$. where $D = \frac{d}{dx}$

$\Rightarrow$ Given :- $(D^2+4)y = x^2+\cos 2x$

$\Rightarrow f(D)y = x^2+\cos 2x$

when $f(D) = D^2+4$

The A.E is

$$f(m) = 0$$
$$m^2+4 = 0$$
$$m^2 = -4$$
$$m = \pm 2i$$

$\therefore y_c = c_1\cos 2x + c_2 \sin 2x$

$$y_p = \frac{x^2+\cos 2x}{D^2+4}$$

$$y_p = \frac{x^2}{D^2+4} + \frac{\cos 2x}{D^2+4}$$

$$y_p = \frac{x^2}{4\left(1+\frac{D^2}{4}\right)} + \frac{x}{2\times 2}\sin 2x$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2 + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{8}+ - - - - +\right)x^2 + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(x^2 - \frac{2}{4}\right) + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{x}{4}\sin 2x$$

$$y_p = \frac{x^2}{4} - \frac{1}{8} + \frac{x}{4}\sin 2x$$

$\therefore y = y_c + y_p$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x^2}{4} - \frac{1}{8} + \frac{x}{4}\sin 2x$$

Method of Variation of parameters

Step 1 :- write the given DE $p_0 \frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = Q(x)$ for constant p_0, p_1, p_2

Step 2 :- write the same in the form of $f(D)y = Q(x)$ and find its complimentary function as $y_c = c_1 y_1 + c_2 y_2$

Step 3 :- write the soln for the given DE by replacing c_1 & c_2 by A & B, we get $y = Ay_1 + By_2$ and where

$$A = -\int \frac{y_2 Q(x)}{w} dx + k_1$$

$$B = \int \frac{y_1 Q(x)}{w} dx + k_2$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

13) Solve $\frac{d^2y}{dx^2} + y = \sec x . \tan x$ using the method of variation of parameter

$\Longrightarrow$ Given:- $\frac{d^2y}{dx^2} + y = \sec x . \tan x$

$\Rightarrow (D^2+1)y = \sec x . \tan x$

$\Rightarrow f(D)y = \sec x . \tan x$

when $f(D) = D^2+1$

the A.E is $f(m) = 0$

$m^2+1=0$

$m^2=-1$

$m = 0 \pm 1i$

$\therefore y_c = c_1\cos x + c_2\sin x$

the soln is $y_c = Ay_1 + By_2$

$y_1 = \cos x \Longrightarrow y_1' = -\sin x$

$y_2 = \sin x \Rightarrow y_2' = \cos x$

$W = y_1y_2' - y_2y_1'$

$W = \cos^2 x + \sin^2 x$

$W = 1$

$A = -\int \frac{y_2\, q(x)}{W}\, dx + k_1$ $\qquad$ $B = \int \frac{y_1\, q(x)}{W}\, dx + k_2$

$= -\int \frac{\sin x \cdot \sec x \cdot \tan x}{1}\, dx + k_1$ $\qquad$ $= \int \frac{\cos x \cdot \sec x \cdot \tan x}{1}\, dx + k_2$

$= -\int \frac{\sin x}{\cos x} \cdot \tan x\, dx + k_1$ $\qquad$ $= \int \left(\frac{\cos x}{\cos x} \cdot \tan x\right) dx + k_2$

$= -\int \tan^2 x\, dx + k_1$ $\qquad$ $= \int ((1) \cdot \tan x)\, dx + k_2$

$= -(\sec^2 x - 1)\, dx + k_1$ $\qquad$ $B = \log(\sec x) + k_2$

$A = -\tan x + x + k_1$

$y = Ay_1 + By_2$

$y = (x - \tan x + k_1)\cos x + [\log(\sec x) + k_2]\sin x$

14] Solve $\frac{d^2y}{dx^2} + y = \sec x$

$\Rightarrow$ given :- $\frac{d^2y}{dx^2} + y = \sec x$

$\Rightarrow (D^2+1)y = \sec x$

$\Rightarrow f(D)y = \sec x$

when $f(D) = D^2+1$

The A.E is $f(m)=0$

$$m^2+1=0$$

$$m = 0 \pm 1i$$

$$\therefore y_c = c_1\cos x + c_2\sin x$$

The soln is $y = Ay_1 + By_2$

$$y_1 = \cos x \Rightarrow y_1' = -\sin x$$

$$y_2 = \sin x \Rightarrow y_2' = \cos x$$

$$W = y_1y_2' - y_2y_1'$$

$$W = \cos^2 x + \sin^2 x$$

$$W = 1$$

$$A = -\int \frac{y_2\phi(x)}{W}\,dx + k_1$$

$$= -\int (\sin x \cdot \sec x)\,dx + k_1$$

$$= -\int \frac{\sin x}{\cos x}\,dx + k_1$$

$$= -\int \tan x\,dx + k_1$$

$$= -\log(\sec x) + k_1$$

$$A = \log(\cos x) + k_1$$

$$B = \int \frac{y_1\phi(x)}{W}\,dx + k_2$$

$$= \int \frac{\cos x \cdot \sec x}{1}\,dx + k_2$$

$$= \int \frac{\cos x}{\cos x}\,dx + k_2$$

$$= \int (1)\,dx + k_2$$

$$B = x + k_2$$

$$y = Ay_1 + By_2$$

$$y = (\log(\cos x) + k_1)\cos x + (x + k_2)\sin x$$

$\boxed{15}$ Solve $\dfrac{d^2y}{dx^2} + y = \tan x$

$\Longrightarrow$ Given :- $\dfrac{d^2y}{dx^2} + y = \tan x$

$$\Rightarrow (D^2+1)y = \tan x$$

$$\Rightarrow f(D)y = \tan x$$

When $f(D) = D^2+1$

The A.E is $f(m)=0$

$$m^2+1=0$$
$$m^2=-1$$
$$m = 0 \pm 1i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

The soln is $y = Ay_1 + By_2$

$$y_1 = \cos x \Rightarrow y_1' = \sin x$$
$$y_2 = \sin x \Rightarrow y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1'$$
$$W = \cos x(\cos x) - (-\sin x)(\sin x)$$
$$W = \cos^2 x + \sin^2 x$$
$$W = 1$$

$$A = -\int \frac{y_2 \phi(x)}{W} dx + k_1$$
$$A = -\int (\sin x \cdot \tan x) dx + k_1$$
$$A = -\int \frac{\sin^2 x}{\cos x} dx + k_1$$
$$A = -\int \frac{1 - \cos^2 x}{\cos x} dx + k_1$$
$$A = -\int (\sec x - \cos x) dx + k_1$$
$$A = -\log(\sec x + \tan x) + \sin x + k_1$$

$$B = \int \frac{y_1 \phi(x)}{W} dx + k_2$$
$$B = \int (\cos x \cdot \tan x) dx + k_2$$
$$B = \int \sin x \, dx + k_2$$
$$B = -\cos x + k_2$$

$$y = [-\log(\sec x + \tan x) + \sin x + k_1] \cos x + [-\cos x + k_2] \sin x$$

16 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

$\Rightarrow$ Given:- $\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2\right)y = e^x \tan x$

$\Rightarrow (D^2 - 2D + 2)y = e^x \tan x$

$\Rightarrow f(D)y = e^x \tan x$

when $f(D) = D^2 - 2D + 2$

The A.E is $f(m) = 0$

$m^2 - 2m + 2 = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2 \times 1}$

$\Rightarrow m = \frac{2 \pm \sqrt{-4}}{2}$

$m = 1 \pm i$

$y_c = [C_1 \cos x + C_2 \sin x] e^x$

$y_c = C_1 e^x \cos x + C_2 e^x \sin x$

$y_c = Ay_1 + By_2$

$y_1 = e^x \cos x \Rightarrow y_1' = e^x \cos x - e^x \sin x$

$y_2 = e^x \sin x \Rightarrow y_2' = e^x \sin x + e^x \cos x$

$w = y_1 y_2' - y_2 y_1'$

$w = [e^x \cdot \cos x][e^x \sin x + e^x \cos x] - [e^x \cdot \sin x][e^x \cos x - e^x \sin x]$

$w = e^{2x} [\cancel{\cos x \sin x} + \cos^2 x - \cancel{\cos x \sin x} + \sin^2 x]$

$w = e^{2x}$

$$A = -\int \frac{y_2 \phi(x)}{w} dx + k_1$$

$$= -\int \left(\frac{e^x \sin x \, e^x \tan x}{e^{2x}}\right) dx + k_1$$

$$= -\int \frac{\sin^2 x}{\cos x} dx + k_1$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx + k_1$$

$$= -\int (\sec x - \cos x) dx + k_1$$

$$A = -\log|\sec x + \tan x| + \sin x + k_1$$

$$B = \int \frac{y_1 \phi(x)}{w} dx + k_2$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx + k_2$$

$$B = -[\cos x + k_2]$$

$$y = [-\log(\sec x + \tan x) + \sin x + k_1] e^x \sin x - (\cos x + k_2) e^x \cos x$$

$\boxed{17}$ Solve $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$ using the method of variation parameters

$\Rightarrow$ Given :- $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

$$\Rightarrow (D^2+1) y = \frac{1}{1+\sin x}$$

$$\Rightarrow f(D) y = \frac{1}{1+\sin x}$$

The A.E is $f(m) = 0$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$m = 0 \pm 1i$$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

The soln is

$$y = Ay_1 + By_2$$

$$y_1 = \cos x \Rightarrow y_1' = -\sin x$$

$$y_2 = \sin x \Rightarrow y_2' = \cos x$$

$$w = y_1 y_2' - y_2 y_1'$$
$$w = \cos x(\cos x) - (\sin x)(-\sin x)$$
$$w = \cos^2 x + \sin^2 x$$
$$w = 1$$

$$A = -\int \frac{y_2\, \phi(x)}{w}\, dx + k_1$$
$$= -\int \left(\sin x \cdot \frac{1}{1+\sin x}\right) dx + k_1$$
$$= -\int \frac{\sin x(1-\sin x)}{1-\sin^2 x}\, dx + k_1$$
$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx + k_1$$
$$= +\int \frac{\sin^2 x - \sin x}{\cos^2 x}\, dx + k_1$$
$$= \int (\tan^2 x - \tan x \cdot \sec(x))\, dx + k_1$$
$$= \int (\sec^2 x - 1) - \tan x \cdot \sec x\, dx + k_1$$
$$A = \tan x - x - \sec x + k_1$$

$$B = \int y_1 \frac{\phi(x)}{w}\, dx + k_2$$
$$= \int \frac{\cos x}{(1+\sin x)}\, dx + k_2$$
$$= \int \frac{\cos x(1-\sin x)}{1-\sin^2 x}\, dx + k_2$$
$$= \int \frac{\cos x - \sin x \cdot \cos x}{\cos^2 x}\, dx + k_2$$
$$= \int (\sec x - \tan x)\, dx + k_2$$
$$= \log(\sec x + \tan x) - \log(\sec x)$$
$$B = \log\left[\frac{\sec x + \tan x}{\sec x}\right] + k_2$$

$$y = [\tan x - \sec x - x + k_2]\cos x + \left[\log\left[\frac{\sec x + \tan x}{\sec x}\right] + k_2\right]\sin x$$

18 Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the method of variation of parameter

$\Rightarrow$ Given $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$$\Rightarrow (D^2 - 1)y = \frac{2}{1+e^x}$$
$$\Rightarrow f(D)y = \frac{2}{1+e^x}$$

The A.E is $f(m)=0$

$$m^2-1=0$$

$$m^2=\pm 1$$

$$\therefore \quad y_c = c_1 e^{-x} + c_2 e^{x}$$

$$y = Ay_1 + By_2$$

$$y_1 = e^{-x} \Rightarrow y_1' = -e^{-x}$$

$$y_2 = e^{x} \Rightarrow y_2' = e^{x}$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = e^{-x}(e^{x}) + e^{x}(e^{-x})$$

$$w = 1+1$$

$$w = 2$$

$$A = -\int \frac{y_2\, \phi(x)}{w} dx + k_1$$

$$= -\int \frac{e^{x} \times \frac{2}{1+e^{x}}}{2} dx + k_1$$

$$= -\log[1+e^{x}] + k_1$$

$$A = -\log\left[\frac{1}{1+e^{x}}\right] + k_1$$

$$B = \int \frac{y_1\, \phi(x)}{w} + k_2$$

$$= \int e^{-x} \frac{\frac{2}{1+e^{x}}}{2} dx + k_2$$

$$= \int \frac{e^{-x}}{1+e^{x}} dx + k_2$$

$$= \int \frac{1}{e^{x}(1+e^{x})} dx + k_2$$

$$= \int \frac{e^{x}}{(e^{x})^2(1+e^{x})} dx + k_2$$

$$e^{x} = t$$

$$e^{x} dx = dt$$

$$= \int \frac{1}{t^2(1+t)}\,dt + k_2$$

$$= \int\left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{t+1}\right)dt + k_2$$

$$= \frac{-1}{t} - \log t + \log(t+1) + k_2$$

$$= \frac{1}{-e^x} - \log e^x + \log(e^x+1) + k_2$$

$$= -e^{-x} - \cancel{\log}\,x + \log(1+e^x) + k_2$$

$$B = \log(1+e^x) - e^{-x} - x + k_2$$

$$y = \left[-\log\left(\frac{1}{1+e^x}\right) + k_1\right]e^{-x} + \left[\log(1+e^x) - e^{-x} - x + k_2\right]e^x$$

$\boxed{19}$ Solve by the variation of parameter $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\Rightarrow$ Given:- $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$$\Rightarrow (D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

$$\Rightarrow (D-3)^2 y = \frac{e^{3x}}{x^2}$$

The A.E is $f(m) = 0$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$$y_c = (c_1 + c_2 x)e^{3x}$$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$y = Ay_1 + By_2$$

$$y_1 = e^{3x} \Rightarrow y_1' = 3e^{3x}$$

$$y_2 = xe^{3x} \Rightarrow y_2' = e^{3x} + 3xe^{3x}$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = e^{3x}\left(e^{3x} + 3xe^{3x}\right) - xe^{3x} \cdot 3e^{3x}$$

$$w = e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$w = e^{6x}$$

$$A = -\int \frac{y_2 Q(x)}{w}\, dx + k_1 \qquad B = \int \frac{y_1 Q(x)}{w}\, dx + k_2$$

$$= -\int \frac{xe^{3x}\, e^{3x}/x^2}{e^{6x}}\, dx + k_1 \qquad = \int \frac{e^{3x} \cdot e^{3x}/x^2}{e^{6x}}\, dx + k_2$$

$$= -\int \frac{1}{x}\, dx + k_1 \qquad = \int \frac{1}{x^2}\, dx + k_2$$

$$= -\log x + k_1 \qquad B = -\frac{1}{x} + k_2$$

$$A = -\log x + k_1$$

$$A = -\log(1/x) + k_1$$

$$y = (-\log x + k_1)e^{3x} + \left(\frac{-1}{x} + k_2\right)e^{3x} + 3xe^{3x}$$

Legendre's Linear differential Equation

Let the Equation $a_0(ax+b)^3 \frac{d^3y}{dx^3} + a_1(ax+b)^2 \frac{d^2y}{dx^2} + a_2(ax+b)\frac{dy}{dx} + a_3 y = Q(x) \rightarrow (1)$ is called Legendre's Linear differential Equation for the Constants a_0, a_1, a_2, a_3 a and b of 3rd order

Step 1 :- Take $\log_e(ax+b) = z$

$$\Rightarrow ax+b = e^z$$

$$\Rightarrow ax = e^z - b$$

$$x = \frac{e^z - b}{a}$$

Step 2 :- Write $(ax+b)\frac{dy}{dx} = a_1 Dy$

$$(ax+b)^2\frac{d^2y}{dx^2}=a^2D(D-1)y \text{ and}$$

$$(ax+b)^3\frac{d^3y}{dx^3}=a^3D(D-1)(D-2)y, \text{ where } D=\frac{d}{dz}$$

Simplify the given D.E by substituting the above and solve

Step 3:- finally in the soln replace $z=\log(ax+b)$

Step 4:- If $a=1$ & $b=0$ in the given legendre's equation then

Eq ① $\Rightarrow a_0x^3\frac{d^3y}{dx^3}+a_1x^2\frac{d^2y}{dx^2}+a_2x\frac{dy}{dx}+a_3y=a(x)$ is called Cauchy's linear differential eqn of III rd order

$\boxed{20}$ Solve $(3x+2)^2y''+3(3x+2)y'-36y=8x^2+4x+1$

$\Rightarrow$ Given :- $(3x+2)^2y''+3(3x+2)y'-36y=3x^2+4x+1$

$$\log_e(3x+2)=z$$

$$\Rightarrow 3x+2=e^z$$

$$x=\frac{e^z-2}{3}$$

and $(3x+2)y'=3.Dy$

$$(3x+2)^2y''=3^2D(D-1)y \quad \text{where } D=\frac{d}{dz}$$

$$① \Rightarrow 9D(D-1)y+3.3Dy-36y=8\left[\frac{e^z-2}{3}\right]^2+4\left[\frac{e^z-2}{3}\right]+1$$

$$\Rightarrow [9D^2-9Dy+9Dy-36y]=\frac{8}{9}(e^{2z}-4e^z+4)+\frac{4}{3}(e^z-2)+1$$

$$\Rightarrow [9D^2-9D+9D-36]y=\frac{1}{9}(8e^{2z}-32e^z+32]+12e^z-24+9]$$

$$\Rightarrow 9(D^2-4)y=\frac{1}{9}[8e^{2z}-20e^z+17]$$

$$\Rightarrow (D^2-4)y=\frac{1}{81}[8e^{2z}-20e^z+17] \longrightarrow ①$$

The AE is $f(m)=0$

$$m^2-4=0$$

$$\Rightarrow (m-2)(m+2)=0$$

$$\Rightarrow m=-2,2$$

$$y_c = c_1 e^{-2z} + c_2 e^{2z}$$

$$y_p = \frac{1}{81}\left(\frac{8e^{2z} - 20e^{z} + 17}{D^2 - 4}\right)$$

$$= \frac{1}{81}\left[\frac{8e^{2z}}{D^2-4} - \frac{20e^{z}}{D^2-4} + \frac{17}{D^2-4}\right]$$

$$= \frac{1}{81}\left[\frac{8e^{2z}}{(D-2)(D+2)} - \frac{20e^{z}}{(1-4)} + \frac{17}{(0-4)}\right]$$

$$y_p = \frac{1}{81}\left[\frac{8z}{1!}\frac{e^{2z}}{4} - \frac{20e^{z}}{3} - \frac{17}{4}\right]$$

$\therefore$ $y = y_c + y_p$

$$y = c_1 e^{-2z} + c_2 e^{2z} + \frac{1}{81}\left[2ze^{2z} + \frac{20}{3}e^{z} - \frac{17}{4}\right]$$

$$y = \frac{c_1}{(e^z)^2} + c_2(e^z)^2 + \frac{1}{81}\left[2z(e^z)^2 + \frac{20}{3}e^{z} - \frac{17}{4}\right]$$

$$y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{81}\left[2(3x+2)^2 \cdot \log(3x+2) + \frac{20}{3}(3x+2) - \frac{17}{4}\right]$$

31 Solve $(1+x)^2\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

$\Rightarrow$ Given :- $(1+x)^2\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Let $\log(x+1) = z$

$(x+1)y' = D \cdot y$

$(x+1)^2 y'' = D(D-1)y$ where $D = \frac{d}{dz}$

① $\Rightarrow$ $[D(D-1)y - Dy + y] = 2\sin z$

$\Rightarrow$ $[D^2 - D - D + 1]y = 2\sin z$

$\Rightarrow$ $[D^2 - 2D + 1]y = 2\sin z$

The A.E is $f(m)=0$

$$m^2-2m+1=0$$

$$\Rightarrow (m-1)^2=0$$

$$\Rightarrow m=1,1$$

$$y_c=(c_1+c_2 z)e^z$$

$$y_p=\frac{2\sin z}{D^2-2D+1}$$

$$=\frac{2\sin z}{-1^2-2D+1}$$

$$=\frac{2\sin z}{-2D}$$

$$=-\frac{\sin z}{D}$$

$$=-\int \sin z\,dz$$

$$y_p=\cos z$$

$$y=y_c+y_p$$

$$y=(c_1+c_2 z)e^x+\cos z$$

$$y=[c_1+c_2\log(x+1)](x+1)+\cos[\log(x+1)]$$

$\boxed{22}$ Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=\sin[2\log(1+x)]$

$\Rightarrow$ Given:- $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=\sin[2\log(1+x)]$

Let $\log(1+x)=z$

and $(1+x)\frac{dy}{dx}=D_1 y$

$(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$ where $D=\frac{d}{dz}$

$(1)\Rightarrow D(D-1)y+Dy+y=\sin 2z$

$\Rightarrow [D(D-1)+D+1]y=\sin 2z$

$\Rightarrow (D^2-D+D+1)y = \sin 2z$

$\Rightarrow f(D)y = \sin 2z$

The A.E is $f(m)=0$

$m^2+1=0$

$m = 0 \pm i$

$\therefore y = c_1 \cos z + c_2 \sin z$

$$y_p = \frac{\sin 2z}{f(D)}$$

$$y_p = \frac{\sin 2z}{D^2+1}$$

$$y_p = \frac{\sin 2z}{(-2)^2+1}$$

$$y_p = \frac{-1}{3}\sin 2z$$

$\therefore y = y_c + y_p$

$y = c_1 \cos z + c_2 \sin z - \frac{1}{3}\sin 2z$

$y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)]$

$\boxed{23}$ $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

$\Rightarrow$ Let $\log(2x+3) = z$

$2x+3 = e^z$

$$x = \frac{e^z-3}{2}$$

and $(2x+3)y' = 2 \cdot Dy$

$(2x+3)^2 y'' = 4D(D-1)y$ where $D = \frac{d}{dz}$

$(1) \Rightarrow 4D(D-1)y - 2Dy - 12y = 6\left(\frac{e^z-3}{2}\right)$

$\Rightarrow (4D^2 - 4D - 2D - 12)y = 3(e^z - 3)$

$$\Rightarrow (4D^2-6D-12)y = 3(e^z-3)$$

$$\Rightarrow (2D^2-3D-6)y = \frac{3}{2}(e^z-3)$$

$$f(D)\,y = \frac{3}{2}(e^z-3) \rightarrow (2)$$

The A.E is $f(m)=0$

$$2m^2-3m-6=0$$

$$m = \frac{3 \pm \sqrt{9-4(2)(-6)}}{2(2)}$$

$$m = \frac{3 \pm \sqrt{57}}{4}$$

$$m = \frac{3+\sqrt{57}}{4},\quad m = \frac{+3+\sqrt{57}}{4}$$

$$\therefore\ y_c = c_1 e^{\left(\frac{3-\sqrt{57}}{4}\right)z} + c_2 e^{\left(\frac{3+\sqrt{57}}{4}\right)z}$$

$$y_P = \frac{3}{2}\left[\frac{e^{z}-3}{2D^2-3D-6}\right]$$

$$y_P = \frac{3}{2}\left[\frac{e^z}{2D^2-3D-6} - \frac{3e^{0z}}{2D^2-3D-6}\right]$$

$$y_P = \frac{3}{2}\left[\frac{e^z}{2-3-6} - \frac{3}{(-6)}\right]$$

$$y_P = \frac{3}{2}\left[\frac{1}{2} - \frac{e^z}{7}\right]$$

$$y = y_c + y_P$$

$$y = c_1(2x+3)^{\frac{3-\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3+\sqrt{57}}{4}} + \frac{3}{2}\left[\frac{1}{2} - \frac{e^z}{7}(2x+3)\right]$$

24) Solve $x^2\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) + y = \log x$

$\Rightarrow$ Given:- $x^2\left(\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right) + y = \log x$

Let $\log x = z$
$x = e^z$

and w.k.T

$x^2y'' = D(D-1)y$

$xy' = Dy \quad , D = \frac{d}{dz}$

(1) $\Rightarrow D(D-1)y - Dy + y = z$

$[D^2 - D - D + 1]y = z$

$(D-1)^2 y = z$

$f(D)y = z$

The A.E is $f(m) = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$\therefore y_c = (c_1 + c_2 z)e^z$

$y_p = \frac{z}{(D-1)^2}$

$y_p = \frac{z}{(1-D)^2}$

$= (1-D)^{-2}$

$= (1 + 2D + 3D^2 + \text{-----})z$

$y_p = z + 2$ where $D = \frac{d}{z}$

$\Rightarrow y = y_c + y_p$

$\Longrightarrow y = (c_1 + c_2 z)e^z + z + 2$

$\Longrightarrow y = [c_1 + c_2 \log x]x + (\log x) + 2$ //

25 Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

$\Longrightarrow$ Given :- $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Let $\log x = z$

$x = e^z$

$xy' = D.y$

$x^2y'' = D(D-1)y$ where $D = \frac{d}{dz}$

(1) $\Rightarrow D(D-1)y - 3Dy + 4y = (1+x)^2$

$\Rightarrow (D^2-D)y - 3Dy + 4y = (1+x)^2$

$\Rightarrow (D^2-D-3D+4)y = (1+x)^2$

$\Rightarrow (D^2-4D+4)y = (1+x)^2$

$f(D)y = (1+x)^2$

A.E is $f(m) = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$\therefore y_c = (c_1 + c_2 z)e^{2z}$

$y_p = \frac{(1+e^z)^2}{(D-2)^2}$

$y_p = \frac{1+e^{2z}+2e^{2z}}{(D-2)^2}$

$y_p = \frac{1}{(D-2)^2} + \frac{e^{2z}}{(D-2)^2} + \frac{2e^{2z}}{(D-2)^2}$

$y_p = \frac{e^{2z}}{(D-2)(D+2)} + \frac{2e^{2z}}{(D-2)4} + \frac{1}{(D-2)^2}$

$$W = y_1 y_2' - y_2 y_1'$$
$$W = e^{-z}(-2e^{-z}) - e^{-2z}(-e^{-z})$$
$$W = -2e^{-3z} + e^{-3z}$$
$$W = e^{-3z}$$

$$A = -\int \frac{y_2 Q(x)}{W} dz + k_1 \qquad B = \int y_1 \frac{Q(x)}{W} dz + k_2$$

$$A = -\int \frac{e^{-2z} e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1 \qquad B = \int \frac{e^{-z} e^{e^z}}{-e^{-3z}} dz + k_2$$

$$A = -\int \frac{-e^{-2z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1 \qquad B = -\int \frac{e^{-z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_2$$

$$A = \int \frac{e^{e^z}}{e^{-z}} dz + k_1 \qquad B = -\int e^{2z} e^{e^z} dz + k_2$$

$$A = \int e^{z} e^{e^z} dz + k_1 \qquad B = -\int e^{z} e^{e^z} (e^{z} dz) + k_2$$

$$A = e^{e^z} + k$$

let $e^z = t$

$$e^z dz = dt$$

$$B = -\int t e^{t} dt + k_2$$

$$B = -((t-1)e^{t} + k_2$$

$$B = (1-t)e^{t} + k_2$$

$$B = (1-e^z)e^{e^z} + k_2$$

$$y = [e^{e^z} + k_1]e^{-z} + [(1-e^z)e^{e^z} + k_2]e^{-2z}$$

$$y = \left(\frac{e^{e^z} + k_1}{e^z}\right) + \frac{(1-e^z)e^{e^z} + k_2}{(e^z)^2}$$

$$y = \left[\frac{e^x + k_1}{x}\right] + \left[\frac{(1-x)e^x + k_2}{x^2}\right]$$

$$= \frac{ze^{2z}}{2!} + \frac{2ze^{z}}{1!} + \frac{1}{4}$$

$$y_p = \frac{\log x e^{2z}}{2} + 2\log x e^{z} + \frac{1}{4}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{2z} + \frac{\log x e^{2z}}{2} + 2\log x e^{z} + \frac{1}{4}$$

26 Solve $x^2y'' + 4xy' + 2y = e^x$

$\Rightarrow$ Given :- $x^2y'' + 4xy' + 2y = e^x \longrightarrow (1)$

let $\log x = z$

$x = e^z$

and w.k.T $xy' = D.y$

$x^2y'' = D(D-1)y$ where $D = \frac{d}{dz}$

$(1) \Rightarrow D(D-1)y + 4Dy + 2y = e^{e^z}$

$\Rightarrow (D^2 - D + 4D + 2)y = e^{e^z}$

$\Rightarrow (D^2 + 3D + 2)y = e^{e^z}$

- The A.E is $f(m) = 0$

$m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

$$y_c = c_1 e^{-z} + c_2 e^{-2z}$$

$$y = Ay_1 + By_2$$

$y_1 = e^{-z} \Longrightarrow y_1' = -e^{-z}$

$y_2 = e^{-2z} \Longrightarrow y_2' = -2e^{-z}$

$$W = y_1 y_2' - y_2 y_1'$$
$$= e^{-z}(-2e^{-z}) - e^{-2z}(-e^{-z})$$
$$W = e^{-3z}$$

$$A = -\int \frac{y_2\, Q(x)}{W} dz + k_1$$
$$= -\int \frac{e^{-2x} \cdot e^{e^z}}{e^{-3z}} dz + k_1$$
$$= -\int \frac{-e^{-2z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1$$
$$= \int \frac{e^{e^z}}{e^{-z}} dz + k_1$$
$$= \int e^{z} \cdot e^{e^z} dz + k_1$$
$$A = e^{e^z} + k_1$$

$$B = \int \frac{y_1\, Q(x)}{W} dz + k_2$$
$$= \int \frac{e^{-z} \cdot e^{e^z}}{-e^{-3z}} dz + k_2$$
$$= -\int \frac{e^{-z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_2$$
$$= -\int e^{2z} \cdot e^{e^z} dz + k_2$$
$$= -\int e^{z} \cdot e^{e^z} (e^{z} dz) + k_2$$

Let $e^{z} = t$

$$\Rightarrow e^{z} dz = dt$$
$$= -\int t e^{t} dt + k_2$$
$$= -(t-1)e^{t} + k_2$$
$$= (1-t)e^{t} + k_2$$
$$B = (1-e^{z})e^{e^z} + k_2$$

$$y = [e^{e^z} + k_1]e^{-z} + [(1-e^{z})e^{e^z} + k_2]e^{-2z}$$
$$y = \left[\frac{e^{e^z} + k_1}{e^{z}} + \frac{(1-e^{z})e^{e^z} + k_2}{(e^{z})^2}\right]$$
$$y = \left[\frac{e^{x} + k_1}{x}\right] + \left[\frac{(1-x)e^{x} + k_2}{x^2}\right]$$

27] Solve $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = x + \dfrac{1}{x^2}$

$\Rightarrow$ Given :- $x^2\dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x} \rightarrow (1)$

Let $\log x = z$
$x = e^z$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

$(1) \Rightarrow D(D-1)y - 2y = e^{2z} + \dfrac{1}{e^z}$

$\Rightarrow (D^2 - D)y - 2y = e^{2z} + e^{-z}$

$\Rightarrow (D^2 - D - 2)y = e^{2z} + e^{-z}$

$\Rightarrow f(D)y = e^{2z} + e^{-z}$

The A.E $f(m) = 0$

$\Rightarrow m^2 - m - 2 = 0$

$\Rightarrow m^2 - m - 2m - 2 = 0$

$\Rightarrow m(m+1) - 2(m+1) = 0$

$\Rightarrow (m+1)(m-2) = 0$

$m = -1, \; m = 2$

$\therefore y_c = c_1 e^{-z} + c_2 e^{2z}$

$$y_p = \frac{e^{2z} + e^{-z}}{D^2 - D - 2}$$

$$y_p = \frac{e^{2z}}{(D-2)(D+1)} + \frac{e^{-z}}{(D+1)(D-2)}$$

$$y_p = \frac{z^1}{(D-2)(D+1)} + \frac{e^{-z}}{(D-2)(D+1)}$$

$$y_p = \frac{z^1}{1!}\,\frac{e^{2z}}{3} - \frac{z}{3}e^{-z}$$

$y_p = \frac{z}{3}(e^{2z} - e^{-z})$

$y = y_c + y_p$

$y = c_1 e^{-z} + c_2 e^{2z} + \frac{z}{3}[e^{2z} - e^{-z}]$

$y = \frac{c_1}{x} + c_2 x^2 + \frac{\log x}{3}[x^2 - 1/x]$

$\boxed{28}$ Solve $(x^2D^2 + xD + 9)y = 3x^2 + \sin(3\log x)$, where $D = \frac{d}{dz}$

$\Rightarrow$ Given:- $(x^2D^2 + xD + 9)y = 3x^2 + \sin(3\log x) \longrightarrow (1)$

$f(D)y = 3x^2 + \sin(3\log x)$

when $f(D) = x^2D^2 + xD + 9$

$\log x = z$
$x = e^z$

The. W.K.T

$x^2D^2 = D(D-1)y$

$xD = Dy$

$(1) \Rightarrow D(D-1)y + Dy + 9y = 3e^{2z} + \sin(3\log x)$

$\Rightarrow (D^2 - D + D + 9)y = 3e^{2z} + \sin(3z)$

$\Rightarrow (D^2 + 9)y = 3e^{2z} + \sin(3z)$

The A.E $f(m) = 0$

$m^2 + 9 = 0$

$m^2 = -9$

$m = 0 \pm 3i$

$\therefore y_c = (c_1 \cos 3z + c_2 \sin 3z)$

$y_p = \frac{3e^{2z}}{D^2+9} + \frac{\sin 3z}{D^2+9}$

$= \frac{3e^{2z}}{4+9} - \frac{z}{2\times 3} \cos 3z$

$$y_p = \frac{3e^{2z}}{13} - \frac{z}{6}\cos 3z$$

$$y = y_c + y_p$$

$$y = (c_1\cos 3z + c_2\sin 3z) + \frac{3e^{2z}}{13} - \frac{z}{6}\cos 3z$$

$$y = c_1\cos 3(\log x) + c_2\sin 3(\log x) + \frac{3x^2}{13} - \frac{\log x}{6}\cos 3(\log x)$$

Application of differential Equation

Sq. The differential Eqn of the displacement $x(t)$ of a spring fixed at the upperend a weight at its Lower End is given by $10x\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0$ the weight is pulled down 0.27cm below the Equilibrium position and then released. find the Expression displacement of the weight from its Equilibrium position at any time during its I^{st} upward motion

$\Longrightarrow$ Given :- the displacement x of a spring oscillations and at any time (t)

The relation of a displacement x with a time t as given $10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0$

It is Second order D.E for damped oscillations of an Oscillator for any $T = \frac{d}{dt}$. we have

$$(1) \Rightarrow (10D^2 + D + 200)x = 0$$

$$f(D)x = 0$$

where $f(D) = 10D^2 + D + 200$

the A.E is $f(m) = 0$

$$\Longrightarrow 10m^2 + m + 200 = 0$$

$$m = -1 \pm \sqrt{\frac{1-4(10)(200)}{2(10)}}$$

$$m = -1 \pm \sqrt{\frac{1-8000}{20}}$$

$$m = -1 \pm \frac{\sqrt{-7999}}{20}$$

$$m = \frac{-1 \pm i\,89.4371}{20}$$

$$m = \frac{-1}{20} \pm i\,\frac{89.4371}{20}$$

$$m = -0.05 \pm i\,4.4718$$

$$x(t) = [c_1 \cos(4.4718)t + c_2 \sin(4.4718)t]\, e^{-0.05t} \longrightarrow (2)$$

W.K.T

when time $t = 0 \Rightarrow x = 0$

$$(2) \Rightarrow \quad 0 = c_1(1) + c_2(0)$$

$$c_1 = 0$$

$$(2) \Rightarrow \quad x(t) = c_2 e^{-0.05t} \cdot \sin(4.4718)t$$

put the amplitude $c_2 = 0.25\,cm$

$$x(t) = (0.25)\, e^{-0.05t} \cdot \sin(4.4718)t$$

Be the displacement of spring travelled at any time (t)

30 If an LCR - circuit. The change on a plane of a Condenser is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin pt$, the circuit is tuned to resistance so that $p^2 = \frac{1}{LC}$, If Initially the current (i) and the charge (q) be zero. Show that for small values of $\frac{R}{L}$, the current is in the circuit at time (t) is given by $\frac{Et}{2L}\sin pt$

$\Rightarrow$ Given :- Inductance (L), resistance (R), capacitance (c) and Charge of the battery (q) and given for any $p^2 = \frac{1}{LC} \Rightarrow p = \frac{1}{\sqrt{LC}}$, where, L, R, C, E are Constants

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin pt \longrightarrow (1)$$

Let $D = \frac{d}{dt}$

$\therefore (1) \Rightarrow (LD^2 + RD + \frac{1}{C})q = E\sin pt$

$\Rightarrow f(D)q = E\sin pt$

The A.E $f(m) = 0$

$\Rightarrow Lm^2 + Rm + \frac{1}{c} = 0$

$\Rightarrow CLm^2 + CRm + 1 = 0$

$$m = \frac{-Rc \pm \sqrt{(Rc)^2 - (4)(cL)(1)}}{2(cL)}$$

$$m = \frac{-Rc \pm \sqrt{R^2c^2 - 4cL}}{2cL}$$

$$m = -\frac{Rc}{2cL} \pm \frac{\sqrt{R^2c^2 - 4cL}}{2cL}$$

$$m = -\frac{R}{2L} \pm \sqrt{\frac{R^2c^2}{4c^2L^2} - \frac{4cL}{4c^2L^2}}$$

$$m = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

$$m = \frac{-R}{2L} \pm \sqrt{\frac{1}{4}\left(\frac{R}{L}\right)^2 - p^2}$$

$\therefore \left(\frac{R}{L} \text{ is small}\right)$

$$m = \frac{-R}{2L} \pm \sqrt{0 - p^2}$$

$$-m = \frac{-R}{2L} \pm pi$$

$$\therefore \quad q_c = (c_1 \cos pt + c_2 \sin pt)\, e^{-\frac{Rt}{2L}} \longrightarrow (2)$$

$$PI = \frac{E \sin pt}{LD^2 + RD + \frac{1}{C}}$$

$$PI = \frac{E \sin pt}{L(-p^2) + RD + \frac{1}{C}}$$

$$= \frac{E \sin pt}{-L\left(\frac{1}{LC}\right) + RD + \frac{1}{C}}$$

$$= \frac{E}{R} \frac{\sin pt}{D}$$

$$= \frac{E}{R} \int \sin pt \, dt$$

$$= \frac{-E}{RP} \cos pt$$

$$\therefore q = CF + PI$$

$$q = [c_1 \cos pt + c_2 \sin pt]\, e^{-\frac{RT}{2L}} - \frac{E}{Rp} \cos pt \longrightarrow (3)$$

$$\Longrightarrow q - \left[1 - \frac{Rt}{2L}\right][c_1 \cos pt + c_2 \sin pt] - \frac{E}{Rp} \cos pt \longrightarrow (4)$$

therefore $i(t) = \frac{dq}{dt}$

$$\Rightarrow i(t) = \left[1 - \frac{Rt}{2L}\right]\left[-c_1 p \sin pt + c_2 p \cos pt\right] - \frac{R}{2L}\left[c_1 \cos pt + c_2 \sin pt\right] + \frac{\epsilon}{R} \sin pt \longrightarrow (5)$$

when $t=0 \Rightarrow q=0$

$$\therefore (4) \Rightarrow 0 = c_1 - \frac{\epsilon}{Rp} \Rightarrow c_1 = \frac{\epsilon}{Rp}$$

when $\epsilon=0 \Rightarrow i=0$

$$\therefore (5) \Rightarrow 0 = c_2 p - \frac{Rc_1}{2L}$$

$$\Rightarrow c_2 p = \frac{Rc_1}{2L}$$

$$\Rightarrow c_2 = \frac{Rc_1}{2Lp}$$

$$\Rightarrow c_2 = \frac{R\left[\frac{\epsilon}{Rp}\right]}{2Lp}$$

$$\Rightarrow c_2 = \frac{\epsilon}{2Lp^2}$$

$$\Rightarrow c_2 = \frac{\epsilon}{2L \cdot \frac{1}{LC}}$$

$$\Rightarrow c_2 = \frac{\epsilon c}{2}$$

$$\therefore (5) \Rightarrow i(t) = \left[1 - \frac{Rt}{2L}\right]\left[\frac{-\epsilon}{Rp} p \sin pt + \frac{\epsilon c}{2} p \cos pt\right] - \frac{R}{2L}\left[\frac{\epsilon}{Rp}\cos pt + \frac{\epsilon c}{2} \sin pt\right] + \frac{\epsilon}{R} \sin pt$$

$$\Rightarrow i(t) = \cancel{\frac{-\epsilon}{R} \sin pt} + \cancel{\frac{\epsilon c}{2} p \cos pt} + \frac{\epsilon t}{2L} \sin pt - \frac{\epsilon c R t}{4L} p \cos pt - \cancel{\frac{\epsilon}{2Cp} \cos pt} - \frac{\epsilon RL}{4L} \sin pt + \cancel{\frac{\epsilon}{R} \sin pt}$$

$$\Rightarrow i(t) = \frac{\epsilon t}{2L} \sin pt \quad \left(\frac{R}{L} \text{ is small}\right)$$

Module-3

partial differential Equation

Defination :- If an Equation involves one dependent Variable and its derivative with respect 2 (or) more independent Variable is called partial differential Eqn

Notation :- Suppose $z = f(x,y)$ be a fun the Variables are x and y Then first and second order partial derivaties can be notated by following Symbols

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial^2 x}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial z}{\partial y}$$

1. from the partial differential Eqn by Eliminating the orbitary Constants a and b from $z = (x-a)^2 + (y-b)^2$

$\Rightarrow$ Given :- $z = (x-a)^2 + (y-b)^2 \longrightarrow (1)$

diff (1) w.r.t 'x' partially

$$(1) \Rightarrow \frac{\partial z}{\partial x} = 2(x-a)$$

$$\Rightarrow p = 2(x-a)$$

$$\Rightarrow (x-a) = \frac{p}{2} \longrightarrow (2)$$

diff (1) w.r.t 'y' partially

$$(1) \Rightarrow \frac{\partial z}{\partial y} = 2(y-b)$$

$$\Rightarrow q = 2(y-b)$$

$$\Rightarrow (y-b) = \frac{q}{2} \longrightarrow (3)$$

from (2) (1) and (3)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$\boxed{4z = p^2 + q^2}$$

2) From the PDE by eliminating the parameters a and b from the sphere

$\Rightarrow$ Given :- $(x-a)^2+(y-b)^2+z^2=r^2 \longrightarrow (1)$

Diff w.r.t 'x' partially

$(1) \Rightarrow 2(x-a)+0+2z\frac{\partial z}{\partial x}=0 \qquad \therefore p=\frac{\partial z}{\partial x}$

$\Rightarrow (x-a)+pz=0$

$\Rightarrow (x-a)=-pz \longrightarrow (2)$

diff (1) w.r.t 'y' partially

$(1) \Rightarrow 0+2(y-b)+2z\frac{\partial z}{\partial y}=0$

$\Rightarrow (y-b)+qz=0 \qquad \therefore q=\frac{\partial z}{\partial y}$

$\Rightarrow y-b=-qz \longrightarrow (3)$

from (1) (2) and (3)

$(1) \Rightarrow (-zp)^2+(-qz)^2+z^2=r^2$

$\Rightarrow p^2z^2+q^2z^2+z^2=r^2$

$$\boxed{(1+p^2+q^2)z^2=r^2}$$

3) from the PDE by eliminating the arbitrary constants a and b from $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

$\Longrightarrow$ Given :- $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

diff (1) w.r.t 'x' partially

$\Rightarrow 2(x-a)+0=2z\frac{\partial z}{\partial x}\cot^2\alpha$

$\Rightarrow (x-a)=zp\cot^2\alpha \longrightarrow (1) \qquad \therefore \frac{\partial z}{\partial x}=p$

diff (1) w.r.t 'y' partially

$\Rightarrow 0+2(y-b)=2z\frac{\partial z}{\partial y}\cot^2\alpha$

$\Rightarrow (y-b)=zq\cot^2\alpha \longrightarrow (2) \qquad \therefore \frac{\partial z}{\partial y}=q$

from ① ② and ③

$$① \Rightarrow (z p \cot^2\alpha)^2 + (z q \cot^2\alpha)^2 = z^2 \cot^2\alpha$$

$$z^2 p^2 \cot^4\alpha + z^2 q^2 \cot^4\alpha = z^2 \cot^2\alpha$$

$$z^2 \cot^4\alpha \left(p^2+q^2\right) = z^2 \cot^2\alpha$$

$$p^2+q^2 = \frac{z^2 \cot^2\alpha}{z^2 \cot^4\alpha}$$

$$p^2+q^2 = \frac{1}{\cot^2\alpha}$$

$$\boxed{p^2+q^2 = \tan^2\alpha}$$

4 from the PDE by Eliminating the orbitary Constant a and b from $2z = \frac{x^2}{a^2}+\frac{y^2}{b^2}$

$\Rightarrow$ Given :- $2z = \frac{x^2}{a^2}+\frac{y^2}{b^2} \longrightarrow ①$

diff ① w.r.t to 'x' partially

$$\Rightarrow 2\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\Rightarrow p = \frac{x}{a^2}$$

$$\Rightarrow a^2 = \frac{x}{p} \longrightarrow ②$$

diff ① w.r.t to 'y' partially

$$\Rightarrow 2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow b^2 = \frac{y}{q} \longrightarrow ③$$

from ② and ③

$$① \Rightarrow 2z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$$2z = \boxed{2z = px+qy}$$

5. from the PDE by Elemenating orbitary constants a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Given :- $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$

Diff (1) w.r.t 'x' partially

$$(1) \Rightarrow \frac{2x}{a^2}+0+\frac{2z\frac{\partial z}{\partial x}}{c^2}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{pz}{c^2}=0 \rightarrow (2)$$

$$\Rightarrow \frac{x}{a^2}=-\frac{pz}{c^2}$$

$$\frac{1}{a^2}=-\frac{pz}{c^2x} \longrightarrow (3)$$

Diff w.r.t 'x' partially

$$(2) \Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p\frac{\partial z}{\partial x}+z\frac{\partial p}{\partial x}\right]=0$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p^2+z\frac{\partial^2 z}{\partial x^2}\right]=0$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p^2+rz\right]=0$$

$$\Rightarrow \frac{-pz}{c^2x}+\frac{1}{c^2}\left[p^2+rz\right]=0$$

$$\Rightarrow \frac{-pz}{x}+\left[p^2+rz\right]=0$$

$$\Rightarrow -pz+x(p^2+rz)=0$$

$$\Rightarrow xp^2+\underline{zxr}=pz$$

$$\Rightarrow x\left(\frac{\partial z}{\partial x}\right)^2+zx\left(\frac{\partial^2 z}{\partial x^2}\right)=z\frac{\partial z}{\partial x}$$

6] From the pde by elimenating the orbitary f^{un} $z = f(x^2+y^2)$

$\Rightarrow$ Given :- $z = f(x^2+y^2) \rightarrow ①$

Diff w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial x} = f'(x^2+y^2)(2x)$$

$$p = f'(x^2+y^2)2x \longrightarrow ②$$

diff ① w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial y} = f'(x^2+y^2)(2y)$$

$$q = 2yf'(x^2+y^2) \longrightarrow ③$$

$$② \div ③ \Rightarrow \frac{p}{q} = \frac{2xf'(x^2+y^2)}{2yf'(x^2+y^2)}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = qx$$

$$\boxed{py - qx = 0}$$

7] from the PDE by elimenating the orbitary function $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$$\Rightarrow \quad z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \rightarrow ①$$

Diff w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial x} = 0 + 2f'\left(1/x + \log y\right)\left(-1/x^2\right)$$

$$p = \frac{-2f'}{x^2}\left(\frac{1}{x} + \log y\right) \longrightarrow ②$$

Diff w.r.t 'y' partially

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(1/y\right)$$

$$q-2y = \frac{2f'}{y}\left(\frac{1}{x}+\log y\right)$$

② ÷ ③

$$\frac{p}{q-2y} = \frac{-\frac{2f'}{x^2}\left(\frac{1}{x}+\log y\right)}{\frac{2}{y}f'\left(\frac{1}{x}+\log y\right)}$$

$$\frac{p}{q-2y} = \frac{-y}{x^2}$$

$$\boxed{px^2+(q-2y)y=0}$$

8] $z = f(x+ay) + g(x-ay)$

$\Rightarrow$ Given :- $z = f(x+ay) + g(x-ay) \longrightarrow ①$

$$\frac{\partial z}{\partial x} = f'(x+ay)(1) + g'(x-ay)(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$$

diff w.r.t 'y' partialy

$$\frac{\partial z}{\partial y} = f'(x+ay)(a) + g'(x-ay)(-a)$$

$$\frac{\partial z}{\partial y} = af'(x+ay) - ag'(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\left[f''(x+ay) + g''(x-ay)\right]$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial z^2}{\partial x^2}}$$

9] from pde by Elemanating $z = yf(x) + x\phi(y)$

$\Rightarrow$ given :- $z = yf(x) + x\phi(y) \longrightarrow ①$

diff w.r.t 'x'

$$\frac{\partial z}{\partial x} = yf'(x) + \phi(y) \longrightarrow ②$$

$$\frac{\partial z}{\partial y} = f(x) + x\phi'(y) \longrightarrow ③$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = yf''(x)$$

$$\Rightarrow \quad r = yf''(x) \rightarrow ④$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = x\phi''(y)$$

$$\Rightarrow \quad t = x\phi''(y) \longrightarrow ⑤$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f'(x) + \phi'(y)$$

$$\Rightarrow \quad s = f'(x) + \phi'(y)$$

$$s = \left[\frac{p - \phi(y)}{y}\right] + \left[\frac{q - f(x)}{x}\right]$$

$$s = \frac{x[p - \phi(y)] + y[q - f(x)]}{xy}$$

$$xys = [px - x\phi y] + qy - yf(x)$$

$$xys = px + qy - z$$

$$px - xys + qy - z = \underline{\underline{0}}$$

$z = xf(x+t) + \phi(x+t).$

10 from PDE $z = xf(x+t) + \phi(x+t) \longrightarrow ①$

$\Rightarrow$ Given :- $z = xf(x+t) + \phi(x+t)$

$$\frac{\partial z}{\partial x} = xf'(x+t) + f(x+t) + \phi_1(x+t) \rightarrow ②$$

$$\frac{\partial z}{\partial t} = xf'(x+t) + \phi'(x+t) \longrightarrow ③$$

$$\frac{\partial^2 z}{\partial x^2} = xf''(x+t) + 2f'(x+t) + \phi''(x+t) \longrightarrow ④$$

$$\frac{\partial^2 z}{\partial t^2} = xf''(x+t) + \phi''(x+t) \rightarrow ⑤$$

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)$$

$$\frac{\partial^2 z}{\partial x \partial t} = xf''(x+t) + f'(x+t) + \phi''(x+t) \longrightarrow (6)$$

$$(4)-(6) \Rightarrow \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial t} = f'(x+t) \longrightarrow (7)$$

$$(6)-(5) \Rightarrow \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} = f'(x+t) \longrightarrow (8)$$

from (7) and (8)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$$

11. From the PDE by elemenating the orbitary function from $lx + my + nz = \phi(x^2+y^2+z^2) \rightarrow$

$\Rightarrow$ $lx + my + nz = \phi(x^2+y^2+z^2) \rightarrow (1)$

Diff w.r.t 'x'

$$(1) \Rightarrow l + 0 + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left[2x + 2z\frac{\partial z}{\partial x}\right]$$

$$\Rightarrow l + np = 2(x+pz)\phi'(x^2+y^2+z^2) \longrightarrow (2)$$

Diff w.r.t 'y'

$$(1) \Rightarrow 0 + m + n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left[2y + 2z\frac{\partial z}{\partial y}\right]$$

$$m + nq = \phi_1(x^2+y^2+z^2)\,2[y+qz] \longrightarrow (3)$$

$$(2) \div (3) \Rightarrow \frac{l+np}{m+nq} = \frac{x+pz}{y+qz}$$

$$(l+np)(y+qz) - (m+nq)(x+pz) = 0$$

formation of PDE for the type $\phi(u,v)=0$

Step 1 :- Suppose u.v is zero be an fun in the variables uv and u.v are the functions of x, y. z and z is a function of x and y

2) Differentiate Equation ① partially w.r.t 'x' $\frac{\partial\phi}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0$

$$\Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x} \longrightarrow ②$$

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial \phi} \longrightarrow ③$$

$2 \div 3$

$$\frac{\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}}{\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}}=\frac{-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}}{-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}}$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}=0$$

$\boxed{12}$ from the pDe for $f(x+y+z,\ x^2+y^2+z^2)=0$

$\Rightarrow$ Given:- $f(x+y+z,\ x^2+y^2+z^2)=0$

$$f(u,v)=0$$

$u=x+y+z$

$\frac{\partial u}{\partial x}=1+0+\frac{\partial z}{\partial x}$

$\frac{\partial u}{\partial x}=1+p$

$\frac{\partial u}{\partial y}=0+1+\frac{\partial z}{\partial y}$

$\frac{\partial u}{\partial y}=1+q$

$v=x^2+y^2+z^2$

$\frac{\partial v}{\partial x}=2x+0+2z\,\frac{\partial z}{\partial x}$

$\frac{\partial v}{\partial x}=2(x+zp)$

$\frac{\partial v}{\partial y}=0+2y+2z\,\frac{\partial z}{\partial y}$

$\frac{\partial v}{\partial y}=2(y+zq)$

$\therefore$ the pde is $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=0$

$$\Rightarrow (1+p)\,2(y+pz)-2(x+pz)(1+q)=0$$

$$\Rightarrow (1+p)(y+qz)-(x+pz)(1+q)=0$$

13 from The pdE by Elemenating orbitary function
$f(x^2+y^2, z-xy)=0$

$\Rightarrow$ Given:- $f(x^2+y^2, z-xy)=0$

$f(u,v)=0$ $\qquad v = z-xy$

$u = x^2+y^2$ $\qquad \frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y$

$\frac{\partial u}{\partial x} = 2x$ $\qquad \frac{\partial v}{\partial x} = p-y$

$\frac{\partial u}{\partial y} = 2y$ $\qquad \frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x$

$\frac{\partial v}{\partial y} = q-x$

$\therefore$ The pdE is $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$

$\Rightarrow 2x(q-x) - (p-y)2y$

$\Rightarrow qx - x^2 - py + y^2 = 0$

$\Rightarrow qx - py - (x^2 - y^2) = 0$

14 from the pdE by Elemenating the function from
$f\left(\frac{xy}{z}, z\right)=0$

$\Rightarrow$ Given :- $f\left(\frac{xy}{z}, z\right)=0$ $\qquad v = z$

$\Rightarrow f(u,v)=0$ $\qquad \frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} = p$

$u = \frac{xy}{z}$ $\qquad \frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} = q$

$\frac{\partial u}{\partial x} = y \frac{\partial}{\partial x}\left(\frac{x}{z}\right)$ $\qquad \frac{\partial u}{\partial y} = x \cdot \frac{\partial}{\partial y}\left(\frac{y}{z}\right)$

$= y\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$ $\qquad = x\left[\frac{z - y\frac{\partial z}{\partial y}}{z^2}\right]$

$\frac{\partial u}{\partial x} = \frac{y}{z^2}(z - px)$ $\qquad \frac{\partial u}{\partial y} = \frac{x}{z^2}(z - qy)$

The pde is $u_x v_y - v_x u_y = 0$

$\Rightarrow q\left[\frac{y}{z^2}(z-px)\right] - p\left[\frac{x}{z^2}(z-py\right] = 0$ $\qquad$ 10

$\Rightarrow py(z-px)-px(z-qy)=0$

Soln of PDE by direct Integration

15 Solve PDE by direct integration method $\frac{\partial^2 z}{\partial x \partial t} = e^{-t}\cdot\cos x$ given $z=0$. when $t=0$ & $\frac{\partial z}{\partial t}=0$ when $x=0$

Given :-

$$\frac{\partial^2 z}{\partial x \partial t} = e^{-t}\cos x \longrightarrow (1)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) = e^{-t}\cos x$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial t}\right) = \int e^{-t}\cos x \, dx$$

$$\Rightarrow \frac{\partial z}{\partial t} = e^{-t}\int \cos x \, dx$$

$\partial x \mid \partial t$

$$\Rightarrow \frac{\partial z}{\partial t} = e^{-t}\sin x + f(t) \rightarrow (2)$$

$$\Rightarrow \partial z = [e^{-t}\sin x + f(t)]\, dt$$

$$\Rightarrow \int \partial z = \sin x \int e^{-t} dt + \int f(t)\, dt$$

$$\Rightarrow z = -e^{-t}\sin x + F(t) + g(x) \longrightarrow (3)$$

where $F(t) = \int f(t)\, dt$

given that when $x=0 \Rightarrow \frac{\partial z}{\partial t}=0$

$(2) \Rightarrow 0 = 0 + f(t)$

$f(t)=0$

$\therefore \; F(t) = \int f(t)\, dt = \int 0.\, dt = 0$

$$z = -e^{-t}\sin x + g(x) \longrightarrow (4)$$

where $z=0$

$(4) \Rightarrow \; 0 = -e^{0}\sin x + g(x)$

$\Rightarrow \; 0 = -\sin x + g(x)$

$g(x) = \sin x$

$z = -e^{-t}\sin x + \sin x$

$z = (-1-e^{-t})\sin x$

16 Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

$\Rightarrow$ Given :- $\frac{\partial^3 z}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \cos(2x+3y)$

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \cos(2x+3y)$$

$$\Rightarrow \int \partial\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \int \cos(2x+3y)\,dx$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{1}{2}\int \sin(2x+3y)\,dy + \int f(y)\,dy$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\cos(2x+3y)}{6} + F(y) + g(x)$$

$$\Rightarrow \int \partial z = -\frac{1}{6}\int \cos(2x+3y)\,dx + F y \int dx + \int g(x)\,dx$$

$$\Rightarrow z = -\frac{\sin(2x+3y)}{12} + x F(y) + G(x) + h(y)$$

where $F(y) = \int f(y)\,dy$, $G(x) = \int g(x)\,dx$

17 Solve $\frac{\partial^2 z}{\partial x^2} = xy$, Subject to the Conditions That $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$ and $z=0$ when $x=0$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} = xy$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial x}\right) = \int xy\,\partial x$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \longrightarrow (1)$$

$$\int \partial z = \int \frac{x^2 y}{2}\,dx + \int f(y)\,dx$$

$$z = \frac{x^3 y}{6} + x f(y) + g(y) \longrightarrow (2)$$

$\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$

$$① \Rightarrow \log(1+y) = \frac{y}{2} + f(y)$$

$$\Rightarrow f(y) = \log(1+y) - \frac{y}{2}$$

$$\therefore z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2} + g(y) \longrightarrow ③$$

when $x=0$, $z=0$

$$③ \Rightarrow g(y)=0$$

$$z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2}$$

18. $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ Subject to the condition $\frac{\partial z}{\partial x} = \log_e x$ when $y=1$ & $z=0$ when $x=1$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}\,\partial y$$

$$\Rightarrow \frac{\partial z}{\partial x} = x\int \frac{1}{y}\,\partial y$$

$$\Rightarrow \frac{\partial z}{\partial x} = x\log y + f(x) \longrightarrow ①$$

$$\Rightarrow \int \partial z = \int x\log y\,\partial x + \int f(x)\,\partial x$$

$$z = \log y \int x\,dx + \int f(x)\,dx$$

$$\Rightarrow z = \frac{x^2}{2}\log y + F(x) + g(y) \rightarrow ②$$

where $F(x) = \int f(x)\,dx$

where $y=1 \Rightarrow \frac{\partial z}{\partial x} = \log_e x$

$$① \Rightarrow \log_e x = x\log 1 + f(x)$$

$$\Rightarrow \log x = 0 + f(x)$$

$\Rightarrow f(x) = \log x$

$f(x) = \int f(x)\,dx$

$= \int \log x\,dx$

$= \int 1 \cdot \log x \cdot \partial x$

$= \log x \cdot \int 1 \cdot dx - \int \frac{1}{x} \cdot \int 1\,\partial x\,\partial x$

$= x\log x = \left(\frac{1}{x} \cdot x\right) \partial x$

$\Rightarrow F(y) = x\log x - x$

(2) $\Rightarrow z = \frac{x^2}{2}\log y + \log x - x + g(y) \longrightarrow$ (3)

when $x=1 \Rightarrow z=0$

(3) $\Rightarrow 0 = \frac{1}{2}\log y + 0 - 1 + g(y)$

$\Rightarrow g(y) = 1 - \frac{1}{2}\log y$

$\Rightarrow g(y) = 1 - \log\sqrt{y})$

$$z = \frac{x^2}{2}\log y + x\log x - x + 1 - \log\sqrt{y})$$

19 Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x . \sin y$. for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$. and $z=0$ if y is an add multiple of $\pi/2$ when $y = (2n+1)\pi/2$ for $n = 0, 1, 2, 3 - - - -$]

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x \partial y} = \sin x . \sin y$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x . \sin y$

$\Rightarrow \int \partial \left(\frac{\partial z}{\partial y}\right) = \int \sin x . \sin y\,dx$

$\frac{\partial z}{\partial y} = -\sin y . \cos x + f(y) \longrightarrow$ (1)

$\Rightarrow \int \partial z = -\cos x \int \sin y\,\partial y + \int f(y)\,\partial y$

$\Rightarrow z = \cos x \cos y + f(y) + g(x) \longrightarrow$ (2)

where $x=0 \Rightarrow \frac{\partial z}{\partial y} = -2\sin y$

$① \Rightarrow -2\sin y = -\sin y + f'(y)$

$\Rightarrow f'(y) = -\sin y$

$f(y) = \int f'(y)\,dy$

$\Rightarrow f y = -\int \sin y\,dy = \cos y$

$② \Rightarrow z = \cos x \cos y + \cos y + g(x) \rightarrow ③$

$y = (2n+1)\frac{\pi}{2} \Rightarrow z = 0$

$③ \Rightarrow 0 = \cos x \cdot \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$

$\Rightarrow 0 = 0 + 0 + g(x)$

$g(x) = 0$

$z = \cos x \cos y + \cos y$

$z = \cos y(1 + \cos x)$

Soln of Homogeneous partial differential Equ

20 Solve $\frac{\partial^2 z}{\partial y^2} = z$ given that $y = 0$. $e^x = z$ and $\frac{\partial z}{\partial y} = e^{-x}$

$\Rightarrow$ Given:- $\frac{\partial^2 z}{\partial y^2} = z$

$\Rightarrow \frac{\partial^2 z}{\partial y^2} - z = 0$

$\Rightarrow (D^2 - 1)z = 0$

The A.E is $(m^2 - 1) = 0$

$m = \pm 1$

$\therefore z = f(x)e^{-y} + g(x)e^{y} \rightarrow ①$

$\frac{\partial z}{\partial y} = -f(x)e^{-y} + g(x)e^{y} \longrightarrow ②$

$y = 0 \Rightarrow z = e^x$

$① \Rightarrow e^x = f(x) + g(x) \rightarrow ③$

& $y = 0 \Rightarrow \frac{\partial z}{\partial y} = e^{-x}$

$(2) \Rightarrow e^{-x} = -f(x) + g(x) \rightarrow (4)$

$(3)+(4) = 2g(x) = e^{x} + e^{-x}$

$g(x) = \frac{e^{x}+e^{-x}}{2} = \cosh x$

$(3)-(4) = 2f(x) = e^{x} - e^{-x}$

$f(x) = \frac{e^{x}-e^{-x}}{2} = \sinh x$

$z = e^{-y}\sinh x + e^{y}\cosh x$

21] Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given that when $x=0$, $z=e^{y}$ and $\frac{\partial z}{\partial x} = 1$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} + z = 0$

$(D^2+1)z = 0$

the A.E is $f(m)=0$

$m^2+1=0$

$m = 0 \pm 1i$

$\therefore z = f(y)\cos x + g(y)\sin x$

$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x$

when $x=0 \Rightarrow z = e^{y}$

$(2) \Rightarrow f(y) = e^{y}$

where $x=0 \Rightarrow \frac{\partial z}{\partial x} = 1$

$(3) \Rightarrow 1 = g(y) \Rightarrow g(y) = 1$

$$\boxed{z = e^{y}\cos x + \sin x}$$

22] Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ Given that when $x=0$, $z=0$ and $\frac{\partial z}{\partial x} = a\sin y$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} = a^2 z \rightarrow (1)$

$\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

$(D^2 - a^2)z = 0$

$m = \pm a$

$$z = f(y)e^{-ax} + g(y)e^{ax} \rightarrow (2)$$

$$\frac{\partial z}{\partial x} = a f(y)e^{-ax} + a g(y)e^{ax} \rightarrow (3)$$

when $x=0 \Rightarrow z=0$

$$(2) \Rightarrow f(y)+g(y)=0 \rightarrow (4)$$

$$x=0 \Rightarrow \frac{\partial z}{\partial x} = a\sin y$$

$$(3) \Rightarrow -af(y) + ag(y) = a\sin y$$

$$\Rightarrow -f(y)+g(y) = \sin y \rightarrow (5)$$

$$(4)-(5) \Rightarrow 2g(y) = \sin y$$

$$g(y) = \frac{\sin y}{2}$$

$$\Rightarrow f(y) = -g(y) = \frac{-1}{2}\sin y$$

$$z = \frac{-1}{2}(\sin y)e^{-ax} + \frac{1}{2}(\sin y)e^{ax}$$

$$z = \sin y\left[\frac{e^{ax}-e^{-ax}}{2}\right]$$

$$z = \sin y . \sinh(ax)$$

23 Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ Subject to the condition that $z=1$ and $\frac{\partial z}{\partial x} = y$ when $x=0$

$\Rightarrow$ Given:- $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0 \longrightarrow (1)$

$$(D^2+3D-4)z=0$$

the A.E is $m^2+3m-4=0$

$$m^2+4m-m-4=0$$

$$m(m+4)-1(m+4)=0$$

$$m=-4, m=1$$

$$z = f(y)e^{-4x} + g(x)e^{x} \longrightarrow (3)$$

when $x=0, \Rightarrow z=1$

$$② \Rightarrow f(y)+g(y)=1 \longrightarrow ④$$

when $x=0 \Rightarrow \frac{\partial z}{\partial x}=y$

$$③ \Rightarrow -4f(y)+g(y)=y \longrightarrow ⑤$$

$$④-⑤ \Rightarrow 5f(y)=1-y$$

$$f(y)=\frac{1-y}{5}$$

$$\Rightarrow g(y)=1-f(y)$$

$$\Rightarrow g(y)=1-\frac{1}{5}(1-y)$$

$$g(y)=\frac{5-1+y}{5}$$

$$g(y)=\frac{y+4}{5}$$

$$\boxed{z=\left(\frac{1-y}{5}\right)e^{-4x}+\left(\frac{y+4}{5}\right)e^{x}}$$

Lagrange's partial differential Eqn

Step 1 :- The general form of Lagrange's linear p.E can be defined $Pp+Qq=R$ where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$. P. Q. R are the functions of x. y. z

Step 2 :- Write the Auxillary Equation for the Lagranges Linear p.D.E as $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Step 3 :- Consider the Suitable pairs and solve the Same, the soln be the $u(x,y,z)=c_1$

$v(x,y,z)=c_2$

Step 4 :- Write the final Solution as $\phi(u,v)=c$

24 Solve $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

$\Rightarrow$ Given :- $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

$\Rightarrow xp+yq=z$

$\Rightarrow Pp+Qq=R$

$P=x\ ,\ Q=y\ ,\ R=z$

The A.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Rightarrow \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

case ① $\Rightarrow \frac{dx}{x}=\frac{dy}{y}$

$\Rightarrow \int\frac{1}{x}dx=\int\frac{1}{y}dy$

$\Rightarrow \log x-\log y=\log c_1$

$\Rightarrow \log(x/y)=\log c_1$

$\Rightarrow \frac{x}{y}=c_1$

case ② $\Rightarrow \frac{dy}{y}=\frac{dz}{z}$

$\Rightarrow \int\frac{1}{y}dy=\frac{1}{z}dz$

$\Rightarrow \log y=\log z+\log c_2$

$\Rightarrow \log y=\log(c_2 z)$

$\Rightarrow y=c_2 z$

$\frac{y}{z}=c_2$

The soln is $\phi\left(\frac{x}{y},\frac{y}{z}\right)=c$

25 Solve $(y^2z)p=x^2(zq+y)$

$\Rightarrow$ given :- $(y^2z)p=x^2(zq+y)$

$\Rightarrow y^2zp=x^2zq+x^2y \Rightarrow y^2zp-x^2zq=x^2y$

$\Rightarrow Pp+Qq=R \Rightarrow y^2z=P\ ,\ Q=x^2z\ ,\ R=x^2y$

The A.E

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

Case (1) :- $\frac{dx}{y^2 z} = \frac{dy}{-x^2 z}$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{-x^2}$$

$$\Rightarrow \int \frac{\cancel{dx}}{x^2}\, dx = -\int y^2\, dy$$

$$\Rightarrow \frac{x^3}{3} = -\frac{y^3}{3} + k_1$$

$$\Rightarrow x^3 + y^3 = 3k_1 \longrightarrow (1)$$

Case (2) :- $\frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$

$$\Rightarrow \int y\, dy = -\int z\, dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + k_2$$

$$\Rightarrow y^2 + z^2 = 2k_2$$

$\therefore$ The Soln is $\phi(x^3+y^3,\ y^2+z^2) = c$

$\boxed{26}$ Solve $(y-z)p + (z-x)q = (x-y)$

$\Rightarrow$ Given :- $(y-z)p + (z-x)q = (x-y)$

$$\Rightarrow Pp + Qq = R$$

$$P = y-z,\ Q = z-x,\ R = x-y$$

The A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \longrightarrow (1)$$

case ① :- $\dfrac{dx+dy+dz}{y-z+z-x+x-y} = \dfrac{dx+dy+dz}{0}$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow \int dx + \int dy + \int dz = 0$

$\Rightarrow x+y+z = c_1$

Case ② :- $\dfrac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)} = \dfrac{xdx+ydy+zdz}{xy-zx+yz-xy+zx-yz}$

$\Rightarrow \dfrac{xdx+ydy+zdz}{0}$

$\Rightarrow xdx+ydy+zdz = 0$

$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = 0$

$\Rightarrow \dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = c_2$

$\Rightarrow x^2+y^2+z^2 = 2c_2$

$\therefore$ the Soln is $\phi(x+y+z,\ x^2+y^2+z^2) = c$

27 Solve $(y^2+z^2)p + xyq = xz$

$\Rightarrow$ given :- $(y^2+z^2)p + xyq = xz$

$Pp + Qq = R$

$\therefore P = y^2+z^2,\ Q = xy,\ R = xz$

the A. E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$\dfrac{dx}{y^2+z^2} = \dfrac{dy}{xy} = \dfrac{dz}{xz}$

case① :- $\dfrac{xdx-ydy-zdz}{x(y^2+z^2)-y(xy)-z(xz)} = \dfrac{xdx-ydy-zdz}{xy^2+xz^2+xy^2-x^2}$

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$\Rightarrow \int x\,dx - \int y\,dy - \int z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_1$$

$$\Rightarrow x^2 - y^2 - z^2 = 2c_1$$

Case (2) :- $\frac{dy}{xy} = \frac{dz}{zx}$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz$$

$$\Rightarrow \log y - \log z = \log c_2$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log c_2$$

$$\Rightarrow \frac{y}{z} = c_2$$

$\therefore$ the Soln is $\phi\left(x^2 - y^2 - z^2, \frac{y}{z}\right) = c$

28 Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$\Rightarrow$ Given :- $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$$Pp + Qq = R$$

$$P = xy^2 - xz^2, \quad Q = yz^2 - x^2y, \quad R = z(x^2 - y^2)$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xy^2 - xz^2} = \frac{dy}{yz^2 - x^2y} = \frac{dz}{z(x^2 - y^2)}$$

Case (1) :- $\frac{x\,dx + y\,dy + z\,dz}{x(xy^2 - xz^2) + y(yz^2 - x^2y) + z^2(x^2 - y^2)}$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} = 0$$

$$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = 2c_1$$

case 2 :- $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \int \frac{1}{x}dx + \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow xyz = c_2$$

$$\phi = (x^2 + y^2 + z^2, xyz)$$

$\boxed{29}$ Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$\Rightarrow$ Given :- $x^2(y-z)p + y^2(z-x)q = z^2(x-y) \longrightarrow (1)$

$$Pp + Qq = R$$

$$P = x^2(y-z), \quad Q = (y^2(z-x)), \quad R = z^2(x-y)$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

case (1) :- $\dfrac{\frac{1}{x^2}dx}{y-z} = \dfrac{\frac{1}{y^2}dy}{z-x} = \dfrac{\frac{1}{z^2}dz}{x-y}$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y - z + z - x + x - y}$$

$$\Rightarrow \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = c_1$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = c_1$$

Case ② :- $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + zy - xy + xz - yz}$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_2$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow (xyz) = c_2$$

$$\phi = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right) = c$$

30 Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

$\Rightarrow$ Given :- $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

$$Pp - Qq = R$$

$$P = (x(y^2+z)),\quad Q = y(x^2+z),\quad R = z(x^2-y^2)$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - zy^2 - zx^2 + zy^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - zy^2 - zx^2 + zy^2}$$

$$\Rightarrow \int x\,dx + \int y\,dy - \int z\,dz = c_1$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = c_1$$

$$x^2 + y^2 - 2z = 2c_1$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2}$$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\phi = (x^2 + y^2 - 2z,\ xyz) = c$$

31] Solve $(y+z)p + (z+x)q = x+y$

$\Rightarrow$ Given :- $(y+z)p + (z+x)q = x+y$

$$Pp + Qq = R$$

$$P = (y+z),\ Q = (z+x),\ R = (x+y)$$

The A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$\Rightarrow \frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dy}{y - x} = \frac{dy - dz}{z - y}$$

Case 1 :- $\frac{1}{2}\frac{d(x+y+z)}{(x+y+z)} = -\frac{d(x-y)}{x-y}$

$$\Rightarrow \frac{1}{2}\int \frac{1}{x+y+z}\, d(x+y+z) = -\int \frac{1}{(x-y)}\, d(x-y)$$

$$\Rightarrow \frac{1}{2}\log(x+y+z) = \log(x-y) + \log c_1$$

$$\Rightarrow \log(\sqrt{x+y+z}) = \log(x-y) + \log c_1$$

$$\Rightarrow \log|(x-y)\sqrt{x+y+z}| = \log c_1$$

$$\Rightarrow (x-y)\sqrt{x+y+z} = c_1$$

Case 2 :- $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$$\Rightarrow \int \frac{1}{x-y}\, d(x-y) = \int \frac{1}{z-y}\, d(z-y)$$

$$\Rightarrow \log(x-y) = \log(y-z) - \log c_2$$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right) - \log c_2$$

$$\Rightarrow \frac{x-y}{y-z} = c_2$$

The soln is $\phi\left[(x-y)\sqrt{x+y+z},\ \frac{x-y}{y-z}\right] = c$

$\boxed{32}$ Solve $(x^2-y^2-z^2)p + (2xy)q = 2zx$

$\Rightarrow$ Given :- $(x^2-y^2-z^2)p + (2xy)q = 2zx$

$$P = x^2-y^2-z^2,\quad Q = 2xy,\quad R = 2zx$$

the A. E

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \longrightarrow (1)$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 + xz^2} = \frac{dy}{2xy} = \frac{dz}{2zx}$$

Case ① :- $\dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} = \dfrac{dy}{2xy}$

$$\Rightarrow \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \int \frac{dy}{y}$$

$$\Rightarrow \log|x^2+y^2+z^2| = \log y + \log c_1$$

$$\Rightarrow \log\left|\frac{x^2+y^2+z^2}{y}\right| = \log c_1$$

$$\Rightarrow \frac{(x^2+y^2+z^2}{y} = c_1$$

Case ② :- $\dfrac{dy}{2xy} = \dfrac{dz}{2zx}$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dz}{z}$$

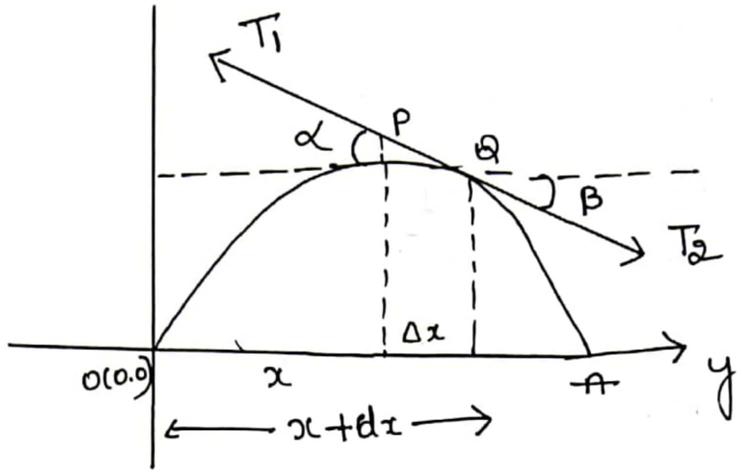
$$\Rightarrow \log y = \log z + \log c_2$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log c_2$$

$$\left(\frac{y}{z}\right) = c_2$$

The sol^n is $\phi\left[\dfrac{x^2+y^2+z^2}{y}, \dfrac{y}{z}\right] = C$

33] One dimensional wave Equation



Consider a Small transverse Vibration of an Elastic string of length l, which is Stretched at the 2 pt origin and A, In the Equilibrium position. OA as the x axis and lined through the origin zero and perpendicular to the x-axis as the y-axis

Let p and q be the 2 points on the string Let (α, β) be the angles at p and q and Let T_1, T_2 be the tension towards the points p and q. Since if There is no motion

$$T_1\cos\alpha = T_2\cos\beta = T \longrightarrow (1)$$

Let m be the mass where Unit length of the string the mass of Element of pq is $m\Delta x$

In the Vertical transverse direction. The Components of T_1 & T_2 are $-T_1\sin\alpha + T_2\sin\beta$

By the newton's Second Law, we have $F = ma \rightarrow (2)$, Here $F = -T_1\sin\alpha + T_2\sin\beta$, The mass $m = m\Delta x$ and a = acceleration $\frac{\partial^2 u}{\partial t^2}$

$$\therefore (1) \Rightarrow T_2\sin\beta - T_1\sin\alpha = m\,dx\,\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{T}{\cos\beta}\sin\beta - \frac{T}{\cos\alpha}\sin\alpha = m\Delta x\,\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow T\tan\beta - T\tan\alpha = m\Delta x\,\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{m\Delta x}{T}\frac{\partial^2 u}{\partial t^2} \longrightarrow (3)$$

Here $\tan\alpha$ & $\tan\beta$ are the slopes at the point p and q

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x=x} \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x=x+\Delta x}$$

$$(3) \Rightarrow \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{m\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} = \frac{m}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} = \frac{m}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{m}{T}\frac{\partial u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

$$\Rightarrow \underbrace{c^2 = \frac{T}{m}}$$

Solution for one dimention Eqn by method of Separation of Variables:-

34) S.T the wave equation for 1-D is $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow$ Given :- $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$

Let the Soln is $u = XT$ where $X = X(x)$, $T = T(t)$

$$\therefore (1) \Rightarrow \frac{\partial^2}{\partial t^2}(XT) = c^2\frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial x}{\partial x^2} = k$$

$$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k, \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k$$

$$\frac{\partial^2 T}{\partial t^2} = kc^2 T, \quad \frac{\partial^2 X}{\partial x^2} = kX$$

$$\Rightarrow \frac{\partial^2 T}{\partial t^2} - kc^2 T = 0 \rightarrow (2) \qquad \frac{\partial^2 X}{\partial x^2} - kX = 0 \longrightarrow (3)$$

case 1) :- If $k = 0$

$$(2) \Rightarrow \frac{\partial^2 T}{\partial t^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}\right) = 0$$

$$\Rightarrow \int \partial\left(\frac{\partial T}{\partial t}\right) = \int 0\, \partial t$$

$$\Rightarrow \frac{\partial T}{\partial t} = c_1$$

$$\Rightarrow \int \partial T = c_1 \int \partial t$$

$$\Rightarrow T = c_1 t + c_2$$

$$(3) \Rightarrow \frac{\partial^2 X}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x}\right) = 0$$

$$\Rightarrow \int \partial\left(\frac{\partial u}{\partial x}\right) = \int 0\, dx$$

$$\Rightarrow \frac{\partial x}{\partial x} = c_1$$

$\Rightarrow \int \partial X = c_1 \int \partial x$

$\Rightarrow X = c_1 x + c_2$

$$\boxed{u = (c_1 x + c_2)(c_1' t + c_2')}$$

case ②:- Let $k = p^2$

② $\Rightarrow \dfrac{\partial^2 T}{\partial t^2} - p^2c^2 T = 0$

$\Rightarrow (D^2 - p^2c^2 T) = 0$, $D = \dfrac{\partial}{\partial t}$

A.E is $f(m) = 0$

$m^2 - p^2c^2 = 0$

$m = \pm pc$

$\therefore T = c_1 e^{-pct} + c_2 e^{pct}$

③ $\Rightarrow \dfrac{\partial^2 X}{\partial x^2} - p^2 X = 0$

$\Rightarrow (D^2 - p^2)X = 0$, $D = \dfrac{d}{\partial x}$

The A E is $m^2 - p^2 = 0$

$\Longrightarrow m = \pm p$

$X = c_1' e^{-px} + c_2 e^{px}$

$u = (c_1' e^{-px} + c_2' e^{px})$

$(c_1 e^{-pct} + c_2 e^{pct})$

case ③:- If $k = -p^2$

② $\Rightarrow \dfrac{\partial^2 T}{\partial t^2} + p^2c^2 T = 0$

$\Rightarrow (D^2 + p^2c^2)T = 0$ $\quad D = \dfrac{\partial}{\partial t}$

The A. E is

$m^2 + p^2c^2 = 0$

$\Rightarrow m = 0 \pm pi$

$\therefore T = c_1 \cos(pct) + c_2 \sin(pct)$

$$③ \Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

$$(D^2+p^2)X = 0 \quad D = \frac{\partial}{\partial x}$$

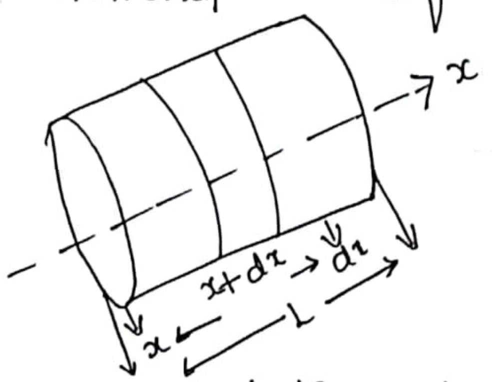
The A.E is $m^2+p^2=0$

$$m = 0 \pm pi$$

$$X = -c_1\cos px + c_2'\sin px$$

$$u = (c_1'\cos px + c_2\sin px)(c_1\cos(pcT) + c_2\sin(pcT))$$

35 One dimentional Heat Equation



Consider a heat conducting Homogeneous rod of length l placed along the x-axis. One End of rod at $x=0$ (origin) Other End of rod at $x=l$. Assume that the rod as constant density ρ and Uniform cross Section A. also assume that the rod is insulated laterally & therefore heat flows Only in the x-direction, let $u(x,t)$ with the temperature of the cross Section at the point x and at any time t The thermal conductivity of k of the material of the rod an The Temperature gradiant $\frac{\partial u}{\partial x}$

q_1 be the Quantity of heat flowing into the cross Section at a distance x in the Unit time is

$q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x s^{-1}$ and q_2 is the Quantity of heat flowing out of the cross Section at a distance $x+dx$

$$q_2 = -KA\left(\frac{\partial x}{\partial x}\right)_{x+\partial x}$$

$$q_1 - q_2 = -KA\left(\frac{\partial u}{\partial x}\right)_x + KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$q_1 - q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \longrightarrow (1)$$

But the rate of Increase of heat in the rod is

$$s\rho A\, dx = \frac{\partial u}{\partial t}$$

$$q_1 - q_2 = s\rho A\, \partial x \frac{\partial u}{\partial t} \longrightarrow (2)$$

s is the specific heat
ρ is density of material

$$s\rho A\, \partial x \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\Rightarrow \frac{s\rho}{K}\frac{\partial u}{\partial t} = \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}$$

$$\Rightarrow \frac{s\rho}{K}\frac{\partial u}{\partial t} = \lim_{\delta x \to 0}\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial x}{\partial x}\right)_x}{\delta x}$$

$$\Rightarrow \frac{s\rho}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{K}{s.\rho}\frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = \frac{K}{s.\rho}$

36) Soln for 1 dimension Heat Eqn by Variable Separation

W.K.T, the one dimensional heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

where u is function of x and t

Let the soln is $V = XT$

where $X = X(x)$, $T = T(t)$

$$(1) \Rightarrow \frac{\partial}{\partial t}(XT) = \frac{c^2 \partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{\cancel{d}T} \frac{\partial T}{\partial t} = \frac{c^2}{X} \frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$$

$$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = K_1, \quad \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$$

$$\Rightarrow \frac{\partial T}{\partial t} = Kc^2 T, \quad \frac{\partial^2 X}{\partial x^2} = KX$$

$$\Rightarrow \frac{\partial T}{\partial t} - Kc^2 T = 0 \longrightarrow (2) \qquad \frac{\partial^2 X}{\partial x^2} - KX = 0 \longrightarrow (3)$$

case 1) If $K = 0$

$$(2) \Rightarrow \frac{\partial T}{\partial t} = 0$$

$$\Rightarrow \int \partial T = \int 0 \, \partial t$$

$$\Rightarrow T = c_1$$

$$(3) \Rightarrow \frac{\partial^2 X}{\partial x^2} = 0 \qquad \text{Let } D = \frac{d}{dx}$$

$$\Rightarrow D^2 X = 0$$

A.E is $m^2 = 0$

$$\Rightarrow m = 0, 0$$

$X_1 = c_2 + c_3 x \qquad \Rightarrow \boxed{u = (c_2 + c_3 x) c_1}$

Case 2) If $K = p^2$

② $\Rightarrow \frac{\partial T}{\partial t} - p^2c^2T = 0$

$\Rightarrow (D - p^2c^2)T = 0$ Let $D = \frac{\partial}{\partial t}$

A.E is $m - p^2c^2 = 0$

$m = p^2c^2$

$T = c_1 e^{p^2c^2t}$

③ $\Rightarrow \frac{\partial^2 X}{\partial x^2} - p^2X = 0$

$\Rightarrow (D^2 - p^2)X = 0$

The A.E is $m^2 - p^2 = 0$

$\Rightarrow m^2 = p^2$

$\Rightarrow m = \pm p$

$X = c_2 e^{-px} + c_3 e^{px}$

$u = (c_2 e^{-px} + c_3 e^{px}) c_1 e^{p^2c^2t}$

Case 3) If $K = -p^2$

② $\Rightarrow \frac{\partial T}{\partial t} + p^2c^2T = 0$

$\Rightarrow (D + p^2c^2)p = 0$

The A.E is $m + p^2c^2 = 0$

$\Rightarrow m = -p^2c^2$

$T = c_1 e^{-p^2c^2t}$

③ $\Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2X = 0$

$\Rightarrow (D^2 + p^2)X = 0$ $D = \frac{\partial}{\partial x}$

The A.E is $m^2 + p^2 = 0$

$m^2 = -p^2$

$m = 0 + pi$

$X = c_2 \cos px + c_3 \sin px$

$$\boxed{u = c_1 e^{-p^2c^2T}[c_2 \cos px + c_3 \sin px]}$$

Module-4

Infinite Series :- If u_n is a function of n, defined for all integral values of n, an expression of the form $u_1+u_2+u_3 - - - u_n +$ --- containing infinite number of terms is called an Infinite Series and is usually denoted by $\sum_{n=1}^{\infty} u_n$ (or) $\sum u_n$

where u_n is the n^{th} term of series (or) the General term of the infinite series.

Suppose if S_n is the sum of the 1st n terms of a series and is denoted by $S_n = u_1+u_2+u_3+ - - - +u_n$

We know that for a geometric series having the 1st term a and the common ratio r. Then the S_n becomes

$$S_n = \frac{a(1-r^n)}{1-r} \quad r<1 \quad \text{and}$$

$$S_n = \frac{a(r^n-1)}{r-1} \quad r>1$$

Note

1. Suppose S_n be the sum of n number of terms of a series. then if $\lim_{n\to\infty} S_n = l$, where l is a finite value. then the given series is called the Convergent and if $\lim_{n\to\infty} S_n=\infty$. Then we say that the given series is a divergent

2.

If Σu_n is the sum of the positive terms (or) a series of positive terms. If $\lim_{n\to\infty} (u_n)^{1/n} = l$ is a infinite and the series is convergent if $l<1$ and divergent if $l>1$ and the test fails if $l=1$

3 a) $\lim_{n\to\infty} n^{1/n} = 1$

b) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2.7$

c) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

1 Test the Convergence of a Series $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+----$

Soln :-

$S_n = 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3} ----$ is a geometric series with common ratio $\frac{1}{3} < 1$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{1\left(1-(1/3)^n\right)}{1-1/3}$$

$$S_n = \frac{\left[1-\frac{1}{3^n}\right]}{2/3}$$

$$S_n = \frac{3}{2}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{3}{2}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2} \lim_{n\to\infty}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2}\left[1-\frac{1}{\infty}\right]$$

$$\Rightarrow \frac{3}{2}(1-0)$$

$$\lim_{n\to\infty} S_n = \frac{3}{2} = \text{finite}$$

Does the Series is Convergent

2] Test the convergence of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---$

Soln Let $S=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---$

Here the n^{th} term of a series is $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$$S_n=\sum u_n$$

$$\Rightarrow S_n=\sum\left|\frac{1}{n}-\frac{1}{n+1}\right|$$

$$\Rightarrow S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+-------$$

$$+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\Rightarrow S_n=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{n(n+1)}{2}$$

$$S_n=\frac{1}{2}\lim_{n\to\infty} n^2\left[1+\frac{1}{n}\right]=\infty$$

Does the series is divergent

3] Test the convergent of $1+2+2^2+2^3+---$

Soln Let $S=1+2+2^2+2^3+---$ is an infinite geometric ratio $r>1$

$$S_n=1+2+2^2+2^3+---+2^n$$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_n=\frac{2^n-1}{2-1}$$

$$S_n=2^n-1$$

$$\lim_{n\to\infty} S_n=\lim_{S\to\infty} 2^n-1=\infty$$

$\therefore$ Does the series is divergent

4) Test The Convergent of $1+2+2^2+2^3+\cdots$

Soln Let $S = 1+2+2^2+2^3+\cdots+2^n$ is an infinite Geometric ratio $r>1$

$$S_n = 1+2+2^2+2^3+\cdots+2^n$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_n = \frac{2^n-1}{2-1}$$

$$S_n = 2^n-1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2^n-1$$

$$= \infty$$

Does the Series is divergent

Cauchy's root test

5) find the nature of the Series $\sum_{n=1}^{\infty} a^{n^2}\cdot x^n$. $a<1$

Soln :- $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} a^{n^2}\cdot x^n \quad a<1$

$$U_n = a^{n^2} x^n$$

$$\Rightarrow (U_n)^{1/n} = (a^{n^2}\cdot x^n)^{1/n} = (a^{n^2})^{1/2}\cdot (x^n)^{1/n}$$

$$\Rightarrow (U_n)^{1/n} = a^n\cdot x$$

$$\Rightarrow \lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty} a^n\cdot x$$

$$\lim_{n\to\infty} U_n^{1/n} = x \lim_{n\to\infty} a^n = x(0) = 0$$

$= \lim_{n\to\infty} U_n = 0<1$ $\therefore$ Does the Series is Convergent

1] Find the nature of the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

Soln :-

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$$

$$\Rightarrow U_n = \left(1+\frac{1}{n}\right)^{n^2}$$

$$\Rightarrow (U_n)^{1/n} = \left|\left(1+\frac{1}{n}\right)^{n^2}\right|^{1/n}$$

$$\Rightarrow (U_n)^{1/n} = \left[1+\frac{1}{n}\right]^{n}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = 2.7 > 1$$

Does the series is divergent

7] test for convergent $\sum_{n=1}^{\infty}\left(n+\frac{1}{n}\right)^{n^2}\cdot\left(\frac{1}{3^n}\right)$

Soln :-

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty}\left|\frac{n+1}{n}\right|^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow U_n = \left(\frac{n+1}{n}\right)^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow U_n = \left(1+\frac{1}{n}\right)^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow (U_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^{n^2}\right]\cdot\frac{1}{3^n}\Bigg]^{1/n}$$

$$\Rightarrow U_n^{1/n} = \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{1/n}\cdot\left[\frac{1}{3^n}\right]^{1/n}$$

$$\Rightarrow U_n^{1/n} = \left(1+\frac{1}{n}\right)^{n}\cdot\frac{1}{3}$$

$$\Rightarrow U_n^{1/n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^{n}$$

$$\lim_{x\to\infty} U_n^{1/n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^{n} = \frac{e}{3} = \frac{2.7}{3} = 0.9 < 1$$

Does the series is convergent

8] Discuss the Convergent $\sum_{n=1}^{\infty} \frac{(n+1)^n \cdot x^n}{n(n+1)}$

Soln :- $U_n = \frac{(n+1)^n \cdot x^n}{n(n+1)}$

$$\Rightarrow U_n = \frac{n^n\left(1+\frac{1}{n}\right)^n \cdot x^n}{n^n \cdot n^1}$$

$$\Rightarrow U_n = \frac{\left(1+\frac{1}{n}\right)^n \cdot x^n}{n}$$

$$\Rightarrow (U_n)^{1/n} = \left[\frac{\left(1+\frac{1}{n}\right)^n \cdot x^n}{n}\right]^{1/n}$$

$$\Rightarrow U_n^{1/n} = \frac{\left[\left(1+1/n\right)^n\right]^{1/n} \cdot (x^n)^{1/n}}{n^{1/n}}$$

$$\Rightarrow U_n^{1/n} = \frac{\left(1+\frac{1}{n}\right) \cdot x}{n^{1/n}}$$

$$\Rightarrow U_n^{1/n} = \frac{x\left(1+\frac{1}{n}\right)}{n^{1/n}}$$

$$\lim_{n\to\infty} U_n^{1/n} = x \cdot \lim_{n\to\infty} \frac{1+1/n}{n^{1/n}} = \frac{x(1+0)}{1} = x$$

$$\lim_{n\to\infty} U_n^{1/n} = x$$

$$\sum U_n = \begin{cases} \text{Convergent if } x<1 \\ \text{Divergent if } x>1 \end{cases}$$

D' alembeart's ratio test

Step 1 :- find the n^{th} term of Series say U_n

Step 2 :- find U_{n+1}

Step 3 :- find $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$ (say)

Step 4 :- if $l<1$, then ΣU_n is convergent

if $l>1$, then ΣU_n is divergent

if $l=1$ Then test fails

9] Test for Convergence the Series $\frac{1^2}{2^1}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}$ $+ - - - -$ $U_n = \frac{n^2}{2^n}$

Soln :-

$$U_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\Rightarrow U_{n+1} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}$$

$$\Rightarrow U_{n+1} = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{n^2(1+1/n)^2}{2^n \cdot 2} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\left(1+\frac{1}{n}\right)^2}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} \lim_{n\to\infty}\left[1+\frac{1}{2}\right]^2 = \frac{1}{2}(1+0) = \frac{1}{2} < 1$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} < 1$$

Does the Series is Convergent

10 Test for the convergent (or) divergent the series $\frac{3}{4+1}+\frac{3^2}{4^2+1}+\frac{3^3}{4^3+1}---$

Soln :-

$$S=\frac{3}{4+1}+\frac{3^2}{4^2+1}+\frac{3^3}{4^3+1}+------$$

The n^{th} Term S is $U_n=\frac{3^{n+1}}{4^{n+1}+1}$

$$U_{n+1}=\frac{3^{n+2}}{4^{n+2}+1}$$

$$\frac{U_{n+1}}{U_n}=\frac{3^{n+2}}{4^{n+2}+1}\times\frac{4^{n+1}+1}{3^{n+1}}$$

$$\frac{U_{n+1}}{U_n}=\frac{3.\,3^{n+1}}{4\cdot4^{n+1}+1}\times\frac{4^{n+1}+1}{3^{n+1}}$$

$$=3\left[\frac{4^{n+1}+1}{4.\,4^{n+1}+1}\right]$$

$$=3\left\{\frac{4^{n+1}\left[1+\frac{1}{4^{n+1}}\right]}{4^{n+1}\left[4+\frac{1}{4^{n+1}}\right]}\right\}$$

$$=3.\left[\frac{1+\frac{1}{4^{n+1}}}{4+\frac{1}{4^{n+1}}}\right]$$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n}=3.\lim_{n\to\infty}\frac{1+\frac{1}{4^{n+1}}}{4+\frac{1}{4^{n+1}}}$$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n}=3\left(\frac{1+0}{4+0}\right)$$

$$\lim_{n\to\infty}\frac{U_{n+1}}{U_n}=\frac{3}{4}<1 \quad \text{Convergent}$$

iii) Discuss the nature of the series $\sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{3}}x^2 + \sqrt{\frac{3}{4}}x^3 + \cdots$

$\Rightarrow$

$$S = \sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}\, x^n$$

$$\therefore\ U_n = \sqrt{\frac{n}{n+1}}\, x^n$$

$$U_{n+1} = \sqrt{\frac{n+1}{n+2}}\, x^{n+1}$$

$$\frac{U_{n+1}}{U_n} = \frac{\frac{\sqrt{n+1}}{\sqrt{n+2}} \times x^{n+1}}{\frac{\sqrt{n}}{\sqrt{n+1}}\, x^n}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot x^n \cdot x^1 \quad \frac{\sqrt{n+1}}{\sqrt{n}} \quad \frac{1}{x^n}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{x^{n+1}}{\sqrt{n(n+2)}}$$

$$= \frac{x \cdot n\left(1+\frac{1}{n}\right)}{n\sqrt{1+\frac{2}{n}}}$$

$$\frac{U_{n+1}}{U_n} = \frac{x\left(1+\frac{1}{n}\right)}{\sqrt{1+\frac{2}{n}}}$$

$$\lt \frac{U_{n+1}}{U_n} = \lt_{n \to x} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{2}{n}}} = x$$

$$\therefore\ \sum U_n = \begin{cases} \text{Convergence for } x < 1 \\ \text{divergence for } x > 1 \\ \text{fail at } x = 1 \end{cases}$$

12. Discuss the nature of the series $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \cdots$

$$\Rightarrow \quad S = \frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \cdots$$

$$S = \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right) x^{n-1}$$

$$U_n = \frac{n}{n+1} x^{n-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^{(n+1)-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^{n}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\frac{n+1}{n+2} x^n}{\frac{n}{n+1} x^{n-1}}$$

$$\Rightarrow \frac{n+1}{n+2} x^n \cdot \frac{n+1}{n} \frac{1}{x^n \cdot x^{-1}}$$

$$\Rightarrow \frac{x(n+1)^2}{n(n+2)}$$

$$\Rightarrow \frac{x n^2 \left[1 + \frac{1}{n}\right]^2}{n^2 \left[1 + \frac{2}{n}\right]}$$

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = x \lim_{x \to \infty} \frac{\left[1 + \frac{1}{n}\right]^2}{\left[1 + \frac{2}{n}\right]}$$

$$\Rightarrow x \left[\frac{(1+0)}{(1+0)}\right] = x$$

$\sum U_n$ = Convergence when $x < 1$

Divergence when $x > 1$

13 Test for convergence (or) divergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

$\Rightarrow$ $$S = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_{n+1} = \frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}}$$

$$U_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\therefore \frac{U_{n+1}}{U_n} = \frac{\dfrac{x^{2n}}{(n+2)\sqrt{n+1}}}{\dfrac{x^{2n-2}}{(n+1)\sqrt{n}}}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n}\, x^{-2}}$$

$$\Rightarrow \frac{x^2(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$\Rightarrow \frac{x^2\sqrt{n+1}\,\cancel{\sqrt{n+1}}\,\sqrt{n}}{(n+2)\,\cancel{\sqrt{n+1}}}$$

$$\frac{U_{n+1}}{U_n} = \frac{x^2\sqrt{n(n+1)}}{n+2}$$

$$= \frac{x^2 n\left(\sqrt{1+\frac{1}{n}}\right)}{n\left(1+\frac{2}{n}\right)}$$

$$= \frac{x^2\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}}$$

$$\lim_{n\to\infty} = x^2 \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}} = x^2$$

$$\sum u_n = \begin{cases} \text{Convergence when } x^2<1 \\ \text{divergence when } x^2>1 \end{cases}$$

Power Series Solution of Second order

Consider a 2nd order differential equation of a form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \longrightarrow (1)$$ where $P_0(x), P_1(x), P_2(x)$ are the polynomial of the x and $P_0(x) \neq 0$ at $x=0$

Step 1 :- Write the Soln of Eqn (1) as a Series

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{----} \rightarrow (2)$$

Step 2 :- find $y' = \sum_{r=1}^{\infty} a_r\, r x^{r-1}$, $y'' = \sum_{r=1}^{\infty} a_r\, r x^{(n-1)(n-2)}$

Substitude the y, y', y'' in Eqn (1) which results in an infinite Series

Step 3 :- In General when the co-efficient of x^r is Equivated To 0, We Obtained recurrence relation, which helps us to determine the Constants a_2, a_3, a_4 ----- in Terms of a_0 & a_1

Step 4 :- thus we get the power Series Solution of the ODE, in The form of $y = a_0 F_1(x) + a_1 F_2(x)$

Note :- Some related Series

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{---}$

2) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \text{---}$

3) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{----}$

4) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

14 Obtain Series Solution of $\frac{d^2y}{dx^2} + y = 0$

Given $\frac{d^2y}{dx^2} + y = 0 \rightarrow (1)$

$P_0(x) = 1 \neq 0$ for $x = 0$, $P_1(x) = 0$. $P_2(x) = 1$

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots \rightarrow (2)$

$y' = \sum_{r=1}^{\infty} a_r r (x)^{n-1} \longrightarrow (3)$

$y'' = \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^r = 0 \rightarrow (4)$

$\therefore$ the co-efficient x^r is $a_{n+2}(r+2)(r+1) + a_r$

$\Rightarrow a_{r+2}(r+1)(r+2) + a_r = 0$

$\Rightarrow (r+1)(r+2)\, a_{r+2} = -a_r$

$a_{r+2} = -\frac{a_r}{(r+1)(r+2)} \rightarrow (5)$

when $r = 0$

$a_2 = -\frac{a_0}{2}$

when $r = 1 \Rightarrow a_3 = \frac{-a_1}{6}$

when $r = 2 \Rightarrow a_4 = \frac{-a_2}{12} = -\frac{(a_1/2)}{12} = \frac{a_0}{24}$

when $r = 3 \Rightarrow a_5 = -\frac{a_3}{20} = -\frac{a_1/6}{20} = -\frac{a_1}{120}$

when $r = 4 \Rightarrow a_6 = \frac{-a_4}{30} = -\frac{a_2/12}{30} = -\frac{a_2}{720}$

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} - \frac{a_0 x^6}{720} + \cdots$$

$$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^5}{120}\right]$$

15 Let us obtain the power series solution of the eqn $\frac{d^2y}{dx^2} - y = 0$

$\Rightarrow$

$$a_{r+2} = \frac{a_r}{(r+2)(r+1)} \quad (r \geq 0)$$

by putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain

$$a_2 = \frac{a_0}{2},\ a_3 = \frac{a_1}{6},\ a_4 = \frac{a_0}{24},\ a_5 = \frac{a_1}{120},\ a_6 = \frac{a_0}{720}$$

Substituting these values in the expended from of $y = \sum_{r=0}^{\infty} a_r x^r$ we obtain

$$y = a_0\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + a_1\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] \longrightarrow (1)$$

we have $(D^2 - 1)y = 0$ where $D = \frac{d}{dx}$

Auxillary Eq is $m^2 - 1 = 0$

$m = \pm 1$

$$y = c_1 e^x + c_2 e^{-x} \longrightarrow (2)$$

$$y = a_0\left[\frac{e^x + e^{-x}}{2}\right] + a_1\left[\frac{e^x - e^{-x}}{2}\right]$$

(or)

$$y = \left[\frac{a_0 - a_1}{2}\right] e^x + \left[\frac{a_0 - a_1}{2}\right] e^{-x}$$

$y = c_1 e^x + c_2 e^{-x}$ where $c_1 = \frac{a_0 + a_1}{2}$

$c_2 = \frac{a_0 - a_1}{2}$

16 Solve $\frac{d^2y}{dx^2}+xy=0$ by obtaining the solution in the form of series

$\Rightarrow$ $P_0(x)=1$, $P_1(x)=0$, $P_2(x)=x$

Let $\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \text{----}$ (2)

$$y' = \sum_{r=1}^{\infty} a_r \cdot r x^{r-1} \longrightarrow (3)$$

$$y'' = \sum_{r=2}^{\infty} a_r \cdot r(r-1) x^{r-2} \longrightarrow (4)$$

$$(1) \Rightarrow \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^r \cdot x$$

$$\sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^{r+1} \longrightarrow (5)$$

The co-efficient x^r is $a_{r+2}(r+2)(r+1) + a_{r-1} \rightarrow (6)$

$$a_{r+2}(r+2)(r+1) = -a_{r-1}$$

$$a_{r+2} = -\frac{a_{r-1}}{(r+2)(r+1)} \longrightarrow (7) \qquad x \geq 1, \; a_2 = 0$$

When $r=1$, $a_3 = -\frac{a_0}{6}$

When $r=2$, $a_4 = -\frac{a_1}{12}$

When $r=3$, $a_5 = -\frac{a_2}{20} = 0$

When $r=4$, $a_6 = -\frac{a_3}{30} = -\frac{a_0/6}{30} = -\frac{a_0}{180}$

Frobenous method

Consider the Second order differential Eqn $p_0(x)\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=0$ for which $p_0(x)=0$ when $x=0$

Working procedure

1) Write the Solution of the Given D.E → (1) as $y=\sum_{r=0}^{\infty} a_r x^{k+r}$ → (2)

$$y=x^k(a_0+a_1x+a_2x^2+a_3x^3+ \dots)$$

2) find $y'=\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$, $y''=\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$ and Substitute the Same in the Equation (1)

3) Equate the co-efficient of x^k [when $r=0$] at 0. we may get the polynomial of Second order in k & Gives two roots, Say k_1 & k_2

4) Also Equate the co-efficient of x^{k+1} [when $r=1$] to 0 and find the value of a_1

5) On both Side Equivate co-efficient of x^{k+r} and get the recurrence relation

6) find the values a_2, a_3, a_3 ---- by The help of The recurrence relation

7) Substitute all the value A is the given Eqn (1) and they $k=k_1$. we make get $y_1(x)$ act $k=k_1$ and Similarly $k=k_2$ we get $y_2(x)$

8) finally write the Soln of given D.E as $y=c_1y_1(x)+c_2y_2(x)$ where c_1 and c_2 are arbitary constants

17 Derivation of bessel's function using frobenous method

⇒ Consider a bessele's differential Eqn

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \longrightarrow (1)$$

$y=0$

Here $P(x)=0$, when $x=0$

∴ the soln is $y = \sum_{r=0}^{\infty} a_r x^{k+r} \longrightarrow (2)$

$$y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \longrightarrow (3)$$

$$y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \longrightarrow (4)$$

Substitude in Eq (1)

$$(1) \Rightarrow x^2 \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + x\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + (x^2-n^2)\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{(k+r)} + \sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} \left[a_r (k+r)(k+r-1) + a_r(k+r) - n^2 a_r\right] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r ((k+r)^2 - n^2) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$a_0(k^2-n^2) = 0$

$a_0 \neq 0$, $k^2-n^2=0$

$k = \pm n$

$k=-n$, $k=+n$

$a_1[(k+1)^2-n^2]=0$

$\Rightarrow (k+1)^2 \neq n^2$

$a_1=0$

By making $a_r[(k+r)^2-n^2]+a_{r-2}=0$

$$\Rightarrow a_r[(k+r)^2-n^2] = -a_{r-2}$$

$$\Rightarrow a_r = \frac{-a_{r-2}}{(k+r)^2-n^2}, \ \forall \ r\geq 2 \rightarrow (6)$$

case (I):- when $k=n$

$$(6) \Rightarrow a_r = \frac{-a_{r-2}}{(n+r)^2-n^2}$$

$$a_r = -\frac{a_{r-2}}{n^2+2nr+r^2-n^2}$$

$$a_r = -\frac{a_{r-2}}{2nr+r^2} \rightarrow (7) \quad \forall \ r\geq 2$$

$$a_2 = -\frac{a_0}{4n+4} = \frac{-a_0}{4(n+1)}$$

$$a_3 = \frac{-a_1}{6n+9} = 0$$

$$a_4 = \frac{-a_2}{8n+16}$$

$$= \frac{-1}{8n+16}\left[-\frac{a_0}{4(n+1)}\right]$$

$$= \frac{a_0}{8(n+2)(4(n+1))}$$

$$a_4 = \frac{a_0}{32(n+1)(n+2)}$$

$$a_5 = \frac{-a_3}{10r+25} = 0$$

$$a_6 = \frac{-a_4}{12(n+3)}$$

$$= \frac{-1}{12(n+3)}\left[\frac{a_0}{32(n+1)(n+2)}\right]$$

$$a_6 = \frac{-a_0}{384(n+1)(n+2)(n+3)}$$

$$(2) \Rightarrow y_1 = x^n\left(a_0 + a_1 x + \frac{-a_0}{4(n+1)}x^2 + 0\,x^3 + \frac{a_0 x^4}{32(n+1)(n+2)} + 0 - \frac{a_0 x^6}{384(n+1)(n+2)(n+3)} + \cdots\right]$$

$$y_1 = a_0 x^n\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)}\right] \rightarrow (8)$$

case (2)

Illy for $k = -n$ we get

$$y_2 = a_0 x^{-n}\left[1 - \frac{x^2}{4(-n+1)} + \frac{x^4}{32(-n+1)(-n+2)} - \frac{x^6}{384(-n+1)(-n+2)(-n+3)} + \cdots\right] \rightarrow (9)$$

$\therefore$ the final soln of the given D.E

$$y = c_1 y_1(x) + c_2 y_2(x)$$

for the standarized of bessel of 1st kind let $a_0 = \frac{1}{2^n\sqrt{1+1}}$

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$$y_1 = a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots\right]$$

Let $y_0 = \dfrac{1}{2^n \Gamma(n+1)}$

$$① \Rightarrow \quad y_1(x) = J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left| 1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots \right|$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{x^2}{4(n+1)(\Gamma(n+1))} + \frac{x^4}{32(n+1)\Gamma(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{x^2}{\Gamma(n+2)} + \frac{x^4}{32\Gamma(n+3)} - \frac{x^6}{384\Gamma(n+4)} + \cdots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0 \left(\frac{x}{2}\right)^{2(0)}}{\Gamma(n+0+1)0} + \frac{(-1)^1 (x/2)^{2(1)}}{\Gamma(n+1+1)\,(1)} + \frac{(-1)^2 (x/2)^{2(2)}}{(\Gamma(n+2+1))\,(2)} + \frac{(-1)^3 (x/2)^{2(3)}}{\Gamma(n+3+1)} + \cdots\right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{n=0}^{\infty} \frac{(-1)^0 (x/2)^{2r}}{\Gamma(n+r+1)\, r}$$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{2r+n}$$

$$\text{Ily}\quad J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\overline{(1-n+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r-n}$$

$$y = A J_n(x) + B J_{-n}(x)$$

18 Show that the Vezuall notation i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$\Rightarrow$ w.k.t

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(n+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r+n} \longrightarrow ①$$

$$n = 1/2$$

$$① \Rightarrow J_{(1/2)}x = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(n_{/2}+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r+1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})}\, r!} \left(\frac{x}{2}\right)^{2r} \left(\frac{x}{2}\right)^{1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})}\, r!} \left(\frac{x}{2}\right)^{2r}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\sqrt{\frac{3}{2}}(0!)} - \frac{1}{\sqrt{(\frac{5}{2})}(1!)} \left(\frac{x}{2}\right)^2 + \frac{1}{\sqrt{(-1/2)}(2!)} \left(\frac{x}{2}\right)^4 - \frac{1}{\sqrt{\frac{9}{2}}(3!)} \left(\frac{x}{2}\right)^6 + \cdots \right]$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{(\frac{1}{2})(\sqrt{1/2})} - \frac{x^2}{(\frac{3}{2})(1/2)(\sqrt{1/2}(4)} + \frac{x^4}{(\frac{5}{2})(\frac{3}{2})(1/2)\sqrt{1/2}(32)} - \frac{x^6}{(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})(1/2)(\sqrt{\frac{1}{2}})384} + \cdots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{3\sqrt{\pi}} + \frac{x^4}{60\sqrt{\pi}} - \frac{x^6}{2520\sqrt{\pi}} + \cdots\right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \times \frac{2}{\sqrt{\pi}}\left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots\right]$$

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}} \times \frac{\sqrt{2}\sqrt{2}}{\sqrt{\pi}} \times \frac{1}{\sqrt{x}\times\sqrt{x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

If $n = -1/2$

$$(1) \Rightarrow J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(-1/2 + r + 1)\, r!}\left(\frac{x}{2}\right)^{2r - 1/2}$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r + 1/2)(r!)}\left(\frac{x}{2}\right)^{2r}\left(\frac{x}{2}\right)^{-1/2}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma\left(r+\frac{1}{2}\right)(r!)}\left(\frac{x}{2}\right)^{2r}$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{x}\left[\frac{1}{\pi/2} - \frac{(x/2)^2}{\left(\Gamma\frac{3}{2}\right)(1)} + \frac{(x/2)^4}{\left(\Gamma\frac{15}{2}\right)(2)} - \frac{(x/2)^6}{\Gamma\left(\frac{7}{2}\right)(6)} + \cdots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\pi/2} - \frac{x^2}{4(1/2)(\pi/2)} + \frac{x^4}{24\sqrt{\pi}} - \cdots\right]$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}} \times \frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Orthogonal properties

Theoram

[19] **Statement** :- If α & β are the distinct root of the Eqn $J_n(x) = 0$, then $\int_0^1 x J_n(\alpha x) \cdot J_n(\beta x) = \begin{cases} 0. & \text{If } \alpha \neq \beta \\ \frac{1}{2}[J_n(\alpha)]^2 & \text{If } (\alpha=\beta) \end{cases}$

Proof :-

W.K.T $J_n(x)$ is the Soln of the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \longrightarrow (1)$$

Ily $J_n(\lambda x)$ is a Soln for

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2(x^2 - n^2)y = 0 \longrightarrow (2)$$

$J_n(\alpha x)$ is a Soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2 x^2 - n^2)y = 0 \longrightarrow (3)$$

& $J_n(\beta x)$ is a Soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0 \longrightarrow (4)$$

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$, then

$$(3) \Rightarrow x^2 u'' + xu' + (\alpha^2 x^2 - n^2)u = 0 \longrightarrow (5)$$

$$(4) \Rightarrow x^2 v'' + xv' + (\beta^2 x^2 - n^2)v = 0 \longrightarrow (6)$$

$$(5)\ \frac{v}{x} \Rightarrow xu''v + vu' + \left(\alpha^2 x - \frac{n^2}{x}\right)uv = 0 \longrightarrow (7)$$

$$(6)\ \frac{u}{x} \Rightarrow xuv'' + uv' + \left(\beta^2 x - \frac{n^2}{x}\right)uv = 0 \longrightarrow (8)$$

$(7)-(8) \Rightarrow x(u''v - uv'') + (vu' - uv') + x(\alpha^2 - \beta^2)uv = 0$

$\Rightarrow x(u''v - uv'') + (vu' - uv') = x(\beta^2 - \alpha^2)uv$

$\Rightarrow \alpha[x(u'v - uv')] = x(\beta^2 - \alpha^2)uv$

$\Rightarrow \int_{x=0}^{1} \alpha(x(vu' - uv')] = (\beta^2 - \alpha^2)\int_0^1 x\,uv\,dx$

$\Rightarrow [x[vu_1 - uv_1]]_0^1 = (\beta^2 - \alpha^2)\int_0^1 x\,uv\,dx$

$\Rightarrow \int_0^1 x J_n(\alpha x) J_n'(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}[J_n(\beta x)\alpha J_n(\alpha x) - \beta J_n(\beta x) J_n(\alpha x)]$

$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}[\alpha J_n(\alpha J_n'(\beta) - \beta J_n(\alpha) J_n'(\beta)]$

If α & β are distinct we have $J_n(\alpha) = 0$, $J_n(\beta) = 0$

$(9) \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$; If $\alpha \neq \beta$

If $\alpha = \beta$ we have $J_n(\alpha) = 0$

$(9) \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{2 J_n'(x) J_n(\beta)}{\beta^2 - \alpha^2}$

$= \lim_{\beta \to \alpha} \frac{\alpha\, J_n(\alpha) - J_n(\beta)}{\beta^2 - \alpha^2} - \frac{0}{0}$

$= \lim_{\beta \to \alpha} \frac{\alpha\, J_n'(\alpha) J_n'(\beta)}{2\beta}$

$= \frac{J_n'(\alpha) J_n'(\beta)}{2\alpha}$

$= \frac{1}{2}[J_n(\alpha)]^2$ if $\alpha = \beta$

Legendre's differential Equation

Sol Consider Legendre's linear differential eqn of 2nd order

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow (1)$$

Let $y = \sum_{r=0}^{\infty} a_r x^r \rightarrow (2)$

$y' = \sum_{r=0}^{\infty} r a_r x^{r-1} \rightarrow (3)$

$y'' = \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} \rightarrow (4)$

$$(1) \Rightarrow (1-x^2)\sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - 2x\sum_{r=0}^{\infty} r a_r x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r =$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1) a_r x^r - \sum_{r=0}^{\infty} 2r a_r x^r + \sum_{r=0}^{\infty} n(n+1) a_r x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \left\{\sum_{r=0}^{\infty} n(r-1) + 2r - n+1)\right\} a_r x^r = 0$$

When $r=0$, the co-efficient of x^{-2} becomes $0(-1)$

$\therefore 0(-1)a_0 = 0$

$0 a_0 = 0$

$\Rightarrow a_0 \neq 0$

When $r=1$ the co-efficient of x^{-1} becomes

$\therefore 1\cdot(0)a_1 = 0$

$\Rightarrow a_1 \neq 0$

Compare the co-efficient of x^r on both sides

$$(5) \Rightarrow \left[(r+2)(r+1)a_{r+2}\right] - \left[r(r-1) + 2r - n(n+1)\right] a_r = 0$$

$$(r+1)(r+2)a_{r+2} - [r^2 - r + 2r - n(n+1)]a^r = 0$$

$$(r+1)(r+2)a_{r+2} - [r(r+1) - n(n+1)a_r = 0$$

$$(r+1)(r+2)a_{r+2} = [r(r+1) - n(n+1)] a_r$$

$$\Rightarrow \quad a_{r+2} = \frac{[r(r+1) - n(n+1)]a^r}{(r+1)(r+2)} \rightarrow (6)$$

when $r = 0$

$$(6) \Rightarrow \quad a_2 = \frac{[0 - n(n+1)]a_0}{2}$$

$$\Rightarrow \quad a_2 = -\frac{n(n+1)}{2} a_0$$

Ily when $r = 1$

$$a_3 = \frac{[2 - n(n+1)]a_1}{6}$$

$$a_3 = \frac{[-n(n+1) - 2]a_1}{6}$$

$$a_3 = -\frac{[n^2 + n - 2]a_1}{6}$$

$$\Rightarrow \quad a_3 = -\frac{[n^2 - n + 2n - 2]a_1}{6}$$

$$= -\frac{[n(n+1) + 2(n-1)]a_1}{6}$$

$$a_3 = -\frac{(n-1)(n+2)a_1}{6}$$

when $r = 2$

$$(6) \Rightarrow \quad a_4 = \frac{[-6 - n^2 - n]a_2}{12}$$

$$a_4 = -\frac{[n^2-2n+3n-6]a_2}{12}$$

$$= -\frac{[n(n-2)+3(n-2)]a_2}{12}$$

$$= -\frac{(n-2)(n+3)a_2}{12}$$

$$= -\left[\frac{(n-2)(n+3)}{12}\right]\left[-\frac{n(n+1)}{2}\right]a_0$$

$$a_4 = \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

When $r=3$

$$a_5 = \frac{[12-n^2-n]a_3}{20}$$

$$a_5 = \frac{(-n^2+n-12)}{20}a_3$$

$$a_5 = -\frac{(n-3)(n+4)}{20}$$

$$= \left[\frac{(n-3)(n+4)}{20}\right]\left[-\frac{(n-1)(n+2)a_1}{6}\right]$$

$$a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{120}a_1$$

(2) $\Rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

$\Rightarrow y = a_0 + a_1x - \frac{n(n+1)}{2}a_0x^2 - \frac{(n-1)(n-2)}{6}a_1x^3 + \frac{n(n+1)(n-2)(n-3)}{24}$

$a_0x^4 + \frac{(n-1)(n-3)(n+2)(n+4)}{120}a_1x^5 + \cdots$

$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right]$

$+ a_1\left[x - \frac{(n-1)(n+2)}{4!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5}x^5\right]$

$y = a_0 y_1(x) + a_1 y_2(x)$

Legendre's differential Equation lead into polynomial

22. W.K.T the legendre's differential eq $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow (1)$ By Taking the Soln $y = \sum_{r=0}^{\infty} a_r x^r$ we have the recurrance relation

$$a_{r+2} = \frac{n(n+1) - r(r+1)}{(r+1)(r+2)} a_r \rightarrow (2)$$

$y = a_0 y_1(x) + a_1 y_2(x)$ for a_0 It may be observed that the polynomial $y_1(x) \cdot y_2(x)$ contained alternative power of x and general form the polynomial represents either of them increasing (or) decreasing in the powers of x and can be represented in the form

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} - - - x \rightarrow (3)$$

where $f(x) = a_0$ for n is Even

$a_1 x_1$ for n is add

When $r = n-2$

$$(2) \Rightarrow a_n = -\frac{n(n+1) - (n-2)(n-1)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{(n^2+n-n^2+3n-2)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{(4n-2)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{2(2n-1)}{n(n-1)} a_{n-2}$$

$$\Rightarrow a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

when $r = n-4$

$$② \Rightarrow a_{n-2} = -\left[\frac{n(n+1)-(n-4)(n-3)}{(n-3)(n-2)}\right] a_{n-4}$$

$$a_{n-2} = -\left[\frac{n^2+n-n^2+7n-12}{(n-2)(n-3)}\right] a_{n-4}$$

$$\Rightarrow a_{n-2} = \frac{[7n-12]}{(n-2)(n-3)}\, a_{n-4}$$

$$\Rightarrow a_{n-2} = \frac{-4(2n-3)}{(n-2)(n-3)}\, a_{n-4}$$

$$\Rightarrow a_{n-2} = -\frac{(n-2)(n-3)}{4(2n-3)}\, a_{n-2}$$

$$a_{n-4} = \left[\frac{-(n-2)(n-3)}{4(2n-3)}\right]\left[\frac{-n(n-1)}{2(2n-1)}\, a_n\right]$$

$$a_{n-4} = \frac{n(n-1)(n-2)(n-3)\, a_n}{8(2n-1)(2n-3)}$$

$$③ \Rightarrow y = f(x) = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} a_n x^{n-4} + - - -$$

$$y = f(x) = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} x^{n-4} - - - -\right] G(x) \rightarrow ④$$

where $g(x) = \begin{cases} a_0 \,(a_n & \text{for } n \text{ is Even} \\ a_1(x)\, a_n & \text{for } n \text{ is add} \end{cases}$

If the constant small a^n is choosen such that $y=f(x)$ becomes 1 when $x=1$ the polynomial so obtained are called Legender's polynomial

denoted by $P_n(x)$

Let us choose $a_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{n!}$ the Eq ① becomes

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} x^{n-4} + \cdots \right.$$

for $n = 0, 1, 2, 3 \cdots$ we can get the polynomials as

$$P_0(x) = 1 \rightarrow (A)$$

$$P_1(x) = x \rightarrow (B)$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1) \rightarrow (C)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \rightarrow (D)$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2+3) \rightarrow (E)$$

in otherwards

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$(C) \Rightarrow 3x^2-1 = 2P_2(x)+1$$

$$\Rightarrow 3x^2 = \tfrac{1}{3}[(2P_2(x)+1]$$

$$\Rightarrow x^2 = \tfrac{1}{3}[2P_2(x) + P_0(x)]$$

$$(D) \Rightarrow 5x^3 - 3x = 2P_3(x)$$

$$\Rightarrow x^3 - 2P_3(x) + 3x$$

$$\Rightarrow x^3 = \tfrac{1}{5}[2P_3(x) + 3P_1(x)]$$

$\textcircled{E} \Rightarrow 35x^4 - 30x^2 + 3 = 8P_4(x)$

$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$

$\Rightarrow x^4 = \frac{1}{35}\left[8P_4(x) + 30\frac{1}{3}\left(2P_2(x) + P_0(x)\right) - 3P_0(x)\right]$

$\Rightarrow x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$

Rodrigue's formula

24 for positive value of in the radrigue formula for legendre polynomials can be defined as

$$P_1(n) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_0(x) = \frac{1}{1.1}(x^2-1) = 0$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{2x}{2} = x$$

$\boxed{23}$ Show that $P_3\cos\theta = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

$\Rightarrow$ $P_3(x) = \frac{1}{2}[5x^3 - 3x] \longrightarrow ①$

$x = \cos\theta$

$① \Rightarrow P_3(\cos\theta) = \frac{1}{2}\left(5(\cos\theta)^3 - 3(\cos\theta)\right) \longrightarrow ②$

W.K.T $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$

$② \Rightarrow P_3(\cos\theta) = \frac{1}{2}\left[\frac{5}{4}\cos 3\theta + 3\cos\theta\right] - 3\cos\theta$

$\Rightarrow P_3(\cos\theta) = \frac{1}{2}\left[\frac{5\cos 3\theta + 3\cos\theta}{4} - 12\cos\theta\right]$

$P_3\cos\theta = \frac{1}{8}\left[5\cos 3\theta + 3\cos\theta\right]$

24 Use Rodrigue's formula S.T $P_4(\cos\theta) = \frac{1}{64}[35\cos\theta + 20\cos\theta + 9]$

W.K.T $\quad P_4(4) = \frac{1}{64}[35x^4 - 30x^2 + 3)$

$x = \cos\theta$

$$P_4(\cos\theta) = \frac{1}{8}\left[35\cos^4\theta - 30\cos^2\theta + 3\right]$$

$$= \frac{1}{8}\left[35(\cos^2\theta)^2 - 30\cos^2\theta + 3\right]$$

$$\cos^2\theta = \left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1}{8}\left[35\left(\frac{1+\cos 2\theta}{2}\right)^2 - 30\left(\frac{1+\cos 2\theta}{2}\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) - \frac{30}{2}(1+\cos 2\theta) + 3\right]$$

$$= \frac{1}{32}\left[35 + 70\cos 2\theta + 35\cos^2\theta - 60 - 60\cos 2\theta + 12\right]$$

$$= \frac{1}{32}\left[35\cos^2 2\theta + 10\cos 2\theta - 13\right]$$

$$= \frac{1}{32}\left[35\left(\frac{1+\cos 4\theta}{2}\right) + 10\cos 2\theta - 13\right]$$

$$= \frac{1}{32}\left[\frac{35 + 35\cos 4\theta + 20\cos 2\theta - 26}{2}\right]$$

$$= \frac{1}{64}\left[35\cos 4\theta + 20\cos 2\theta + 9\right]$$

25 Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomial

$$\Rightarrow f(x) = x^3 + 2x^2 - x - 3(1)$$

W.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$f(x) = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right] + \frac{2}{3}\left[2P_2(x) + P_0(x)\right] - P_1(x) - 3P_0(x)$$

$$= \frac{1}{15}\left[3\left(2P_3(x) + 3P_1(x)\right) + 10\left[2P_2(x) + P_0(x)\right] - 15P_1(x) - 45P_0(x)\right]$$

$$\Rightarrow f(x) = \frac{1}{15}\left[6P_3(x) + 9P_1(x) + 20P_2(x) + 10P_0(x) - 15P_1(x) - 45P_0(x)\right]$$

$$f(x) = \frac{1}{15}\left[6P_3(x) + 20P_2(x) - 6P_1(x) - 35P_0(x)\right]$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x)$$

26 Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre's polynomial

$$\Rightarrow f(x) = x^4 + 3x^3 - x^2 + 5x - 2(1) \longrightarrow (1)$$

W.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$$

$$(1) \Rightarrow f(x) = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right] + \frac{3}{5}\left[2P_3(x) + 3P_1(x)\right]$$
$$- \frac{1}{3}\left[2P_2(x) + P_6(x)\right] + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{105}\Big\{3\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right] + 63\left[2P_3(x) + 3P_1(x)\right]$$
$$- 35\left[2P_2(x) + P_0(x)\right] + 525P_1(x) - 210P_0(x)\Big\}$$

$$= \frac{1}{105}\Big[24P_4(x) + 60P_2(x) + 20P_0(x) + 126P_3(x) + 184P_1(x)$$
$$- 70P_2(x) - 35P_0(x) + 525P_1(x) - 210P_0(x)\Big]$$

$$\Rightarrow f(x) = \frac{1}{105}\Big[24P_4(x) + 126P_3(x) - 10P_2(x) + 74P_1(x) -$$
$$224P_0(x)\Big]$$

$$f(x) = \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{32}{5}P_0(x)$$

27) If $x^3 + 2x^2 - x + 1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$ find the values of a.b.c.d

Let $f(x) = x^3 + 2x^2 - x + 1$

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \left[\frac{1}{3}\left\{(2P_2(x) + P_0(x)\right\}\right]$$

$$x^3 = \frac{1}{5}\left[2P_2(x) + 3P_1(x)\right]$$

$$① \Rightarrow f(x) = \frac{1}{5}[2P_3(x) + 3P_1(x)] + \frac{2}{3}[2P_2(x) + P_0(x)] - P_1(x) + P_0(x)]$$

$$= \frac{1}{15}\{3[2P_3(x) + 3P_1(x)] + 10[2P_2(x) + P_0(x)] - 15P_1(x) + 15P_0(x)\}$$

$$= \frac{1}{15}[6P_3(x) + 9P_1(x) + 20P_2(x) + 10P_0(x) - 15P_1(x) + 15P_0(x)]$$

$$= \frac{1}{15}[25P_0(x) - 6P_1(x) + 20P_2(x) + 6P_3(x)]$$

$$aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x) = \frac{5}{3}P_0(x) + \left(-\frac{2}{5}\right)P_1(x) + \left(\frac{4}{3}\right)P_2(x) + \left(\frac{2}{5}\right)P_3(x)$$

$$a = \frac{5}{3},\ b = -\frac{2}{5},\ c = \frac{4}{3},\ d = \frac{2}{5}$$

29) Express $f(x) = x^3 - 5x^2 + 14x + 5$ in terms of legendre's polynomial

$$f(x) = x^3 - 5x^2 + 14x + 5(1) \longrightarrow ①$$

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$$

$$① \Rightarrow f(x) = \frac{1}{5}[2P_3(x) + 3P_1(x)] - \frac{5}{3}[2P_2(x) + P_0(x)] + 14P_1(x) + 5P_0(x)$$

$$= [6P_3(x) + 9P_1(x) - 50P_2(x) - 25P_0(x) + 210P_1(x) + 75P_0(x)]$$

$$= \frac{1}{15}[6P_3(x) - 50P_2(x) + 219P_1(x) + 50P_0(x)]$$

$$\frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{73}{5}P_1(x) + \frac{10}{3}P_0(x)$$

25) Express $f(x) = 3x^3 - x^2 + 5x - 2(1)$ in terms of legender's polynomial

$$\Longrightarrow f(x) = 3x^3 - x^2 + 5x - 2(1)$$

$$1 - P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$f(x) = \frac{3}{5}\left[2P_3(x) + 3P_1(x)\right] - \frac{1}{3}\left[2P_2(x) + P_0(x)\right] + 5P_1(x) \rightarrow 2P_0(x)$$

$$= \frac{3}{5}\left[2P_3(x) + 3P_1(x)\right] - \frac{1}{3}\left[2P_2(x) + P_0(x)\right] + 5P_1(x) - 2P_0(x)$$

$$= \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{15}\left[18P_3(x) + 27P_1(x) - 10P_2(x) - 5P_0(x) + 75P_1(x) - 30P_0(x)\right]$$

$$= \frac{1}{15}\left[-35P_0x + 102P_1(x) - 10P_2(x) + 18P_3(x)\right]$$

$$= \frac{-35}{15}P_0(x) + \frac{102}{15}P_1(x) - \frac{10}{15}P_2(x) + \frac{18}{15}P_3(x)]$$

$$= \frac{-7}{3}P_0(x) + \frac{34}{5}P_1(x) - \frac{2}{3}P_2(x) + \frac{6}{5}P_3(x)$$

Module - 5

Numerical methods

Divided differences :- Suppose $y=f(x)$ be a function in the Variable of x. Let the set of values of y are value of y are $y_0, y_1, y_2, \ldots y_n$ corresponding to the value of an arguments $x_0, x_1, x_2, x_3, \ldots x_n$ the divided difference are classified into 2 types in the Equal length of Interval

1. forward divided difference
2. Backward divided difference

Forward divided differences :- the Symbol Δ (delta) is called the forward divided differences operator and $\Delta^1, \Delta^2, \Delta^3 \ldots$ are called the 1st, 2nd, 3rd order divided forward differences

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1-y_0=\Delta y_0$		
x_1	y_1		$\Delta y_1-\Delta y_0=\Delta^2 y_0$	
				$\Delta^2 y_1-\Delta^2 y_0=\Delta^3 y_0$
x_2	y_2	$y_2-y_1=\Delta y_1$		
			$\Delta y_2-\Delta y_1=\Delta^2 y_1$	
x_3	y_3	$y_3-y_2=\Delta y_2$		$\Delta^2 y_2-\Delta^2 y_1=\Delta^3 y_1$
			$\Delta y_3-\Delta y_2=\Delta^2 y_2$	
x_4	y_4	$y_4-y_3=\Delta y_3$		

$\Delta^4 y$ $\quad \Delta^3 y_1-\Delta^3 y_0=\Delta^4 y_0$

Here $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ is called leading forward divided differences

Backward differences :- the Symbol dell (∇) is called the Backward divided differences Operator. $\nabla, \nabla^2, \nabla^3$ ---- are called the 1st, 2nd, 3rd ---- Order backward divided differences

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$y_1 - y_0 = \nabla y_1$			
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
x_3	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
		$y_4 - y_3 = \nabla y_4$			
x_4	y_4				

Here $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n$ --- are called the leading Backward divided differences

Interpolation and Extrapolation

the Evaluation of y in the given range x_0 to x_n is called called in interpolation and Outside of x_0 to x_n is called Extrapolation

Interpolation formula

i) $y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$
$+$ ------ when $p = \frac{x - x_0}{h}$ is called the newton's forward interpolation formula

ii) $y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$

---- where $p = \frac{x - x_n}{h}$ is called the newton's Backward Interpolation formula

1 The population of a town is given by the table

year	1951	1961	1971	1981	1991
population	19.6	39.65	58.81	72.21	94.61

Using newtons forward and Backward interpolation formula. Calculate the Increase in population from the year 1955 to 1985

year	population (y)	I DD	II DD	III DD	IV DD
$x_0 = 1951$	$y_0 = 19.6$				
		20.05			
$x_1 = 1961$	$y_1 = 39.65$		−0.89		
		19.16		−4.87	
$x_2 = 1971$	$y_2 = 58.81$		−5.76		19.63
		13.4		−14.76	
$x_3 = 1981$	$y_3 = 72.21$		9		
		22.4			
$x_4 = 1991$	$y_4 = 94.61$				

To find y(1955)

Since 1955 is near to $x_0 = 1951$

$\Rightarrow p = \frac{x - x_0}{h}$

$\Rightarrow p = \frac{1955 - 1951}{10}$

$\Rightarrow p = \frac{4}{10}$

$\Rightarrow p = 0.4$

$\therefore$ By the newtons forward intercolation formula

$$\Rightarrow y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\Rightarrow y(1955) = 19.6 + (0.4)(20.05) + \frac{(0.4)(-0.6)(-0.84)}{2} + \frac{(0.4)(0.6)(-1.6)(-4.8}{6}$$

$$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)(19.63)}{24}$$

$$\Rightarrow y(1955) = 19.6 + 8.02 + 0.1008 - 0.31168 - 0.8166$$

$$\Rightarrow y(1955) = 26.60$$

To find $y(1985)$

Since 1985 is near to $x_0 = 1951$

$$p = \frac{x - x_0}{h} = \frac{1985 - 1991}{10} = \frac{-6}{10} = -0.6$$

$\therefore$ The newtons Backward intercolation formula

$$\Rightarrow y(x) = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 +$$

$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_4$$

$$\Rightarrow y(1985) = 94.61 + (-0.6)(22.4) + \frac{(-0.6)(0.4)(9)}{2} +$$

$$\frac{(-0.6)(0.4)(1.4)(14.76)}{6} + \frac{(-1.6)(0.4)(1.4)(2.4)(19.6)}{24}$$

$$\Rightarrow y(1985) = 94.61 - 13.44 - 1.88 - 0.826 - 0.659$$

$$\Rightarrow y(1985) = 78.605$$

x.y from population between 1955 to 1985

$$\Longrightarrow 78.605 - 26.60$$

$$\Longrightarrow 52.005 \,//$$

2) Use an appropriate interpolation formula to compute g(42) using the following data

x	40	50	60	70	80	90
y	184	204	226	250	276	304

x	y	I DD	II DD	III DD	IV DD	V
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

To find y(42)

$y(42)$ is near to $x_0 = 40$

$$p = \frac{x - x_0}{h} = \frac{42-40}{10} = \frac{2}{10} = 0.2$$

By newtone forward interpolation formula

$$\Rightarrow y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 \text{-----}$$

$$\Rightarrow y(42) = 184 + (0.2)(20) + \frac{(0.2)(0.8)(2)}{(2)}$$

$$\Rightarrow y(42) = 184 + 4 - 0.16$$

$$\Rightarrow y(42) = 187.84$$

3 Use an appropriate interpolation formula to compute y(82) and y(92) for the data

x	80	85	90	95	100
y	5026	5674	6362	7088	7854

x	y	I DD	II DD	III DD	IV DD
80	5026				
		648			
85	5674		40		
		688		−2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

i) To find y(82)

Since 82 is near to $x_0 = 80$

$$p = \frac{x - x_0}{h} = \frac{82 - 80}{5} = \frac{2}{5} = 0.4$$

By the newtons forward interpolation formula

$$\Rightarrow y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\Rightarrow y(82) = 5026 + (0.4)(648) + \frac{(0.4)(-0.6)(40)}{2} + \frac{(0.4)(-0.6)(-1.6)(-2)}{6} + \frac{4 \times 0.4(-0.6)(-1.6)(-2.6)}{24}$$

$$\Rightarrow y(82) = 5026 + 259.2 - 4.8 - 0.128 - 0.1664$$

$$\Rightarrow y(82) = 5280.0056$$

ii) To find y(92)

Since 92 is near to $x_4 = 90$

$$p = \frac{x - x_0}{h} = \frac{92-100}{5} = \frac{-8}{5} = -1.6$$

By the newtons Backward Interpolation formula

$$\Rightarrow y(x) = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \text{------}$$

$$\Rightarrow y(92) = 7854 + (-16)(766) + \frac{(-1.6)(-0.6)(40)}{2} + \frac{(-1.6)(-0.6)(0.4)(2)}{6}$$

$$+ \frac{(-1.6)(-0.6)(0.4)(1.4)(4)}{24}$$

$$\Rightarrow y(92) = 7854 - 1225 + 19.2 + 0.128 + 0.0896$$

$$\Rightarrow y(92) = 6647.8176$$

4 Using find $f(12.5)$ for The following data (NBIF)

x	10	11	12	13
y	22	24	28	34

x	y	I DD	II DD	III DD
10	22			
		2		
11	24		2	
		4		0
12	28		2	
		6		
13	34			

To find $y(12.5)$

$$p = \frac{x - x_4}{h} = \frac{-13 + 12.5}{1} = -0.5$$

$$y(x) = y_4 + p\Delta y_4 + \frac{p(p+1)}{2!}\Delta^2 y_4 + \text{-----}$$

$$y(12.5) = (34) + (-0.5)(6) + \frac{(-0.5)(0.5)}{2} 2 + \text{---}$$

$$\Rightarrow \quad y(12.5) = 34 - 3 - 0.25$$

$$\Rightarrow \quad y(12.5) = \underline{\underline{30.75}}$$

5 ~~formula~~ ~~following~~ table and Estimate the number of Students who obtained marks between 40 and 45

marks	30-40	40-50	50-60	60-70	70-80
No of Students	31	42	51	35	31

The number of Student who obtained $<40 = 31$, $<50 = 73$, $<60 = 124$, $<70 = 154$, $<80 = 190$

x	y	I DD	II DD	III DD	IV DD
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		51			
80	190				

x	$y=f(x)$	I DD	II DD	III DD
x_0	$f(x_0)$			
		$\dfrac{f(x_1)-f(x_0)}{x_1-x_0}=f(x_0,x_1)$		
x_1	$f(x_1)$		$\dfrac{f(x_1,x_3)-f(x_0,x_3)}{x_3-x_0}=f(x_0,x_1,x_2)$	
		$\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=f(x_1,x_2)$		
x_2	$f(x_2)$		$\dfrac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}=f(x_1,x_2,x_3)$	
		$\dfrac{f(x_3)-f(x_2)}{x_3-x_1}=f(x_2,x_3)$		
x_3	$f(x_3)$			

$$\frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}=f(x_0,x_1,x_2,x_3)$$

$$f(x)=f(x_0)+(x-x_0)f(x_0-x_1)+(x-x_0)(x-x_1)f(x_0x_1x_2)+(x-x_0)(x-x_1)(x_1-x_2)f(x_0,x_1,x_2,x_3)$$

6. Using newtons interpolation formula to construct the polynomial by the following data

x	2	4	5	6	8	10
$f(x)$	10	96	196	350	868	1746

x	$f(x)$	I DD	II DD	III DD	IV DD	V DD
2	10					
		43				
4	96		19			
		100		2		
5	196		27		0	
		154		2		0
6	350		35		0	
		259		2		
8	868		45			
		439				
10	1746					

To find $y(45)$

Since 45 is near to $x_0 = 40$

$$p = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$$

By the newton's forward interpolation formula

$$\Rightarrow y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \text{-----------}$$

$$\Rightarrow y(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2!}(9) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{6} + \frac{(0.5)(-0.5)(-1.5)(-2.5)(37)}{24}$$

$$\Rightarrow y(45) = 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$\Rightarrow y(45) = 47.687 \simeq 48$$

The Number of students who obtained less than 45 marks = 48

The Number of students who obtained between 40 and 45 marks = 48 - 31 = 17

Divided difference for Unequal intervals

Newton's divided differences formula for Unequal intervals :-

Suppose $y = f(x)$ be a function in x and $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$ ---- be the Values of $f(x)$ Corresponding to the Values of x_0, x_1, x_2, x_3 ---- with Unequal intervals

7. find the Interpolating polynomial

X	0	1	2	3	4	5
f(x)	3	2	7	24	59	118

x	$f(x)$	I DD	II DD	III DD	IV DD	V
0	3					
		-1				
1	2		3			
		5		1		
2	7		6		0	
		17		1		0
3	24		9		0	
		35		1		
4	59		12			
		59				
5	118					

$$f(x) = f(x_0) + (x-x_0) f(x_0 - x_1) + (x-x_0)(x-x_1) + f(x_0, x_1, x_2) +$$

$$------$$

$$f(x) = 3 + (-1)(x-0) + 3(x-0)(x-1) + 1(x-0)(x-1)(x-2)$$

$$f(x) = 3 - x + 3x^2 - 3x + x^3 - 3x^2 + 2x$$

$$f(x) = x^3 - 2x + 3$$

when $x = 6$

$$f(6) = (6)^3 - (2 \times 6) + 3$$

$$f(6) = \underline{\underline{207}}$$

8. Find the number of men getting wages below Rs 3.5 from the following data

Wages in Rs	0-10	10-20	20-30	30-40
frequency	9	30	35	42

The number of men who getting less than 10 = 9

$< 20 = 30$

$< 30 = 35$

$< 40 = 42$

x	y	I DD	II DD	III DD
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

35 is near to $x_3 = 40$

By newton Backward Integrated factor

$$p = \frac{x - x_3}{h} = \frac{35-40}{10} = -0.5$$

$$y(x) = y_3 + p \nabla y_3 + \frac{p(p+1)}{2!} \nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_3$$

$$y(35) = 116 + (-0.5)42 + \frac{(-0.5)(0.5)7}{2} + \frac{2(-0.5)(0.5)(1.5)}{6}$$

$$y(35) = 116 - 21 - 0.875 - 0.125$$

$$y(35) = \underline{\underline{94}}$$

9) Determine the Interpolating formula construct the polynomial for the following data hence find

x	3	7	9	10
$f(x)$	168	120	72	63

Soln

x	$f(x)$	$f(x_0,x_1)$	$f(x_0,x_1,x_2)$	$f(x_0,x_1,x_2,x_3)$
3	168			
		-12		
7	120		-2	
		-24		1
9	72		5	
		-9		
10	63			

$$\Rightarrow f(x) = 168 - 12(x-3) - 2(x-3)(x-7) + 1(x-3)(x-7)(x-9)$$

$$\Rightarrow f(x) = 168 - 12x + 36 - 2x^2 + 20x - 42 + x^3 - 16x + 63x - 3x^2 + 48x - 189$$

$$\Rightarrow f(x) = x^3 - 21x^2 + 119x - 27$$

$$\Rightarrow f(8) = 8^3 - 21(8^2) + 119(8) - 27$$

$$\Rightarrow f(8) = 93$$

10) From the table of half year premium for policy measuring of different states estimate the premium for policy measuring on the age of (46)

age	45	50	55	60	65
premium	114.84	96.16	83.32	74.48	68.48

age (x)	premium	I DD	II DD	III DD	IV DD
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

To find f(46) it is near to $x_0 = 45$

$$p = \frac{x - x_0}{h} = \frac{46-45}{5} = 0.2$$

By newton forward integrated factor

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$y(46) = 114.84 + (0.2 \times -18.68) + \frac{(0.2)(-0.8)}{2} \times 5.84 + \frac{(0.2)(-0.8)(-1.8)(-1.84)}{6} + \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24} \times 0.68$$

$$y(46) = 114.84 - 3.736 - 0.4672 - 0.08832 - 0.022846$$

$$y(46) = 110.5256$$

Lachranges interpolation formula for Unequal Intervals

Suppose $y_0, y_1, y_2, y_3, \ldots, y_n$ be the set of values of $y = f(x)$ corresponding to $x_0, x_1, x_2, \ldots, x_n$ then the Lachranges interpolation formula is follows as

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_n-x_0)(x_n-x_1)(x_n-x_2)} \ldots \frac{(x-x_{n-1})}{(x_n-x_{n-1})} y_n$$

Ily the Inverse legranges interpolation formula can be

$$x = g(x) = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots(y_0-y_n)} x_0 +$$

$$\frac{(y-y_0)(y-y_2)(y-y_3)\ldots(y_1-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3\ldots(y_1-y_n)} x_1 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)} x_2 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_3)\ldots(y_n-y_{n-1})} x_n$$

1] Use the legranches interpolation formula to find y at $x = 0$

x	5	6	9	11
y	12	13	14	16

Given :-

x	y
$x_0 = 5$	$y_0 = 12$
$x_1 = 6$	$y_1 = 13$
$x_2 = 9$	$y_2 = 14$
$x_3 = 11$	$y_3 = 16$

To find $y(x) = y(10)$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y(10) = \frac{(10-6)(10-9)(10-11)(12)}{(5-6)(5-9)(5-11)} + \frac{(10-5)(10-0)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$y(10) = \frac{4(1)(-1)(12)}{(-1)(-4)(-6)} + \frac{5(1)(-1)(13)}{(1)(-3)(-5)} + \frac{(5)(4)(-1)(14)}{(4)(3)(-1)} + \frac{(5)(4)(1)(16)}{(6)(5)(2)}$$

$$y(10) = 2 - 4.33 + 11.66 + 5.33$$

$$y(10) = \underline{\underline{14.66}}$$

12 By the polynomial $f(x)$ using langrange's formula from the following data

x	0	1	2	5
y	2	3	12	147

Soln :-

x	y
$x_0 = 0$	$y_0 = 2$
$x_1 = 1$	$y_1 = 3$
$x_2 = 2$	$y_2 = 12$
$x_3 = 5$	$y_3 = 147$

$$f(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$f(x) = \frac{2(x-1)(x-2)(x-5)}{(-1)(-2)(-5)} + \frac{3(x-0)(x-2)(x-5)}{(1)(-1)(-4)} + \frac{12(x)(x-1)(x-5)}{2(1)(-4)} + \frac{147(x-0)(x-1)(x-2)}{5(4)(3)}$$

$$f(x) = \frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}(x-2)(x-5)(x-0) - 2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$$

$$f(x) = \frac{1}{20}\left| -4(x-1)(x^2-7x+10) + 15x(x^2-7x+10) - 40x(x^2-(x-5)) + 49x(x^2-3x+2) \right|$$

$$f(x) = \frac{1}{20}\left| -4x^3+32x^2-68x+40+15x^3-105x^2+150x-40x^3+240x^2-200x+44x^3-147x^2-98x \right|$$

$$f(x) = \frac{1}{20}\left| 20x^3+20x^2-20x+40 \right|$$

$$f(x) = x^3+x^2-x+2 \,//$$

Numerical integration

Suppose $y=f(x)$ be a function and let $y_0 \cdot y_1 \cdot y_2 --- y_n$ are the Set of Values Corresponding to the partition of

$P = [a=x_0,\ x_0+h=x_1,\ x_1+h=x_2,\ x_2+h=x_3+ ------ + x_{n-1}+h=x_n=b]$ with the Equal length of the Cartesian $h=\frac{b-a}{n}$ where

n is the number of Equal Strips [or $n+1$ ordinate] we can Evaluate $\int_a^b f(x)\,dx$ by the following method

i) Simpsons' $\frac{1}{3}$rd rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6 ---- y_{n-2}) + 4(y_1+y_3+y_5 + ----- + y_{n-1})\right]$$

ii) Simpson's $\frac{3}{8}$th rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+ --- y_{n-3}) + 3(y_1+y_2+y_4+y_5 --- -1)\right]$$

iii) weedles rules for $n=6$

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

13) Evaluate $\int_0^1 \frac{1}{(1+x^2)}\,dx$ by taking Equal Strips and Hence diduse an approximate Value of radian π by following method

i) Simpson's $\frac{1}{3}$rd rule

ii) Simpson's $\frac{3}{8}$ rule

iii) Weddle's rule

Soln :- Let $I = \int_0^1 \frac{1}{1+x^2} dx$

$a=0,\ b=1,\ n=6,\quad y = \frac{1}{1+x^2}$

$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

$P = \{a = x_0 = 0,\ x_1 = \frac{1}{6},\ x_2 = \frac{2}{6},\ x_3 = \frac{3}{6},\ x_4 = \frac{4}{6},\ x_5 = \frac{5}{6},$
$x_6 = 1 = b\}$

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = \frac{1}{1+0} = 1$
$x_1 = 1/6$	$y_1 = \frac{1}{1+36} = 0.9730$
$x_2 = 1/3$	$y_2 = \frac{1}{1+\frac{1}{9}} = 0.9$
$x_3 = 1/2$	$y_3 = \frac{1}{1+\frac{1}{4}} = 0.8$
$x_4 = 2/3$	$y_4 = \frac{1}{1+\frac{4}{9}} = 0.6923$
$x_5 = 5/6$	$y_5 = \frac{1}{1+\frac{25}{36}} = 0.5902$
$x_6 = 1$	$y_6 = \frac{1}{1+1^2} = 0.5$

i) Simpson's $\frac{1}{3}$rd rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 - y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2} dx = \frac{1/6}{3}\left[(1+0.5) + 2(0.9+0.6923) + 4(0.97304+0.8+0.5902)\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2} dx = \frac{1}{18}\left[1.5 + 3.1846 + 9.4528\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = 0.7854$$

$$\Rightarrow \tan^{-1}(x)\Big|_0^1 = 0.7854$$

$$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0) = 0.7854$$

$$\Rightarrow \frac{\pi}{4} = 0.7854$$

$$\Rightarrow \pi = 0.7854 \times 4$$

$$\Rightarrow \pi \simeq 3.1416$$

ii) Simpson's $\frac{3}{8}^{th}$ rule

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\Rightarrow \int_a^b \frac{1}{1+x^2}dx = \frac{3(1/6)}{8}\left[(1+0.5) + 2(0.8) + 3(0.973+0.9+0.6923 + 0.5402\right]$$

$$\Rightarrow \frac{1}{16}\left[1.5+1.6+9.4665\right]$$

$$\Rightarrow \tan^{-1}(x)\Big|_0^1 = 0.7854$$

$$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0) = 0.7854$$

$$\Rightarrow \frac{\pi}{4} = 0.7854$$

$$\Rightarrow \pi = 0.7854 \times 4$$

$$\Rightarrow \pi \simeq 3.1416$$

iii) weddle rule

$$\int_a^b f(x)dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_0\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \frac{3(1/6)}{10}\left[1+4.865+0.9+4.8+0.6923+2.95\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \frac{1}{20}\left[15.7083\right]$$

$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0) = 0.7854$

$\Rightarrow \frac{\pi}{4} = 0.7854$

$\Rightarrow \pi \simeq 3.1416$

$\boxed{14}$ Evaluate $\int_0^1 \frac{1}{1+x} dx$ taking 7 ordinates and Hence deduce the value of $\log_e^2$ by using
i) Simpsons

Soln :- Let $a=0$. $b=1$, $n=6$

$h = \frac{b-a}{6} = \frac{1-0}{6} = \frac{1}{6}$

$P = (a = x_0 = 0, \; x_1 = 1/6, \; x_2 = 2/6, \; x_3 = 3/6, \; x_4 = 4/6, \; x_5 = 5/6$
$x_6 = 6/6)$

x	$y = \frac{1}{1+x}$
$x_0 = 0$	$y_0 = 1$
$x_1 = 1/6$	$y_1 = 0.8571$
$x_2 = 1/3$	$y_2 = 0.73$
$x_3 = 1/2$	$y_3 = 0.667$
$x_4 = 2/3$	$y_4 = 0.6$
$x_5 = 5/6$	$y_5 = 0.545$
$x_6 = 1$	$y_6 = 0.5$

i) Simpsons $\frac{1}{3}$rd rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_2 + y_5)\right]$$

$$= \left(\frac{1/6}{3}\right)\left[(1+0.5) + 2(0.75 + 0.6) + 4(0.857 + 0.669 + 0.5455)\right]$$

$$= \frac{1}{18}\left[1.5 + 2.7 + 8.2784\right]$$

$\log_e(x+1)\Big|_0^1 = 0.6932$

$\log_2^e = 0.6932$

15 Find the approximate value of $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ by Simpson's $\frac{1}{3}$rd rule by dividing $[0, \frac{\pi}{2}]$ into 6 equal parts

<u>Given</u> :- $y(\theta) = \sqrt{\cos\theta}$

$$a = 0, \; b = \frac{\pi}{2}, \; n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12} = 15^\circ (0.2619)$$

$$P = \{a = \theta_0 = 0, \; \theta_1 = 15^\circ, \; \theta_2 = 30^\circ, \; \theta_3 = 45^\circ, \; \theta_4 = 60^\circ, \; \theta_5 = 75^\circ$$
$$\theta_6 = 90^\circ = b\}$$

θ	$y = \sqrt{\cos\theta}$
$\theta = 0$	$y = \sqrt{\cos 0} = 1 = y_0$
$\theta_1 = 15$	$y = \sqrt{\cos 15} = 0.982 = y_1$
$\theta_2 = 30^\circ$	$y = \sqrt{\cos 30} = 0.9306 = y_2$
$\theta_3 = 45^\circ$	$y = \sqrt{\cos 45} = 0.8109 = y_3$
$\theta_4 = 60^\circ$	$y = \sqrt{\cos 60} = 0.7071 = y_4$
$\theta_5 = 75^\circ$	$y = \sqrt{\cos 75} = 0.508 = y_5$
$\theta_6 = 90^\circ$	$y = \sqrt{\cos 90} = 0 = y_6$

By the Simpsons $\frac{1}{3}$rd rule
$n = 6$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_5) + 4(y_1 + y_3 + y_5)\right]$$

$$\Rightarrow \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = \frac{0.2619}{3}\left[(1+0) + 2(0.9306 + 0.7071) + 4(0.9828 + 0.8409 + 0.5087)\right]$$

$$\Rightarrow \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = 1.1777$$

10) Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$ taking 7 ordinates using weddle rule. Hence find $\log_e 2$

Given :- $y = \frac{x}{1+x^2}$

$a=0,\ b=1,\ n=6$

$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

$p = \left\{a - x_0 = 0,\ x_1 = \frac{1}{6},\ x_2 = \frac{1}{3},\ x_3 = \frac{1}{2},\ x_4 = \frac{2}{3},\ x_5 = \frac{5}{6},\ x_6 = 1 = b\right\}$

x	$\frac{x}{1+x^2}$
0	$0 = y_0$
1/6	$0.1621 = y_1$
1/3	$0.30 = y_2$
1/2	$0.4 = y_3$
2/3	$0.4615 = y_4$
5/6	$0.4915 = y_5$
1	$0.5 = y_6$

By The weddle rule

$$\int_0^b f(x)\,dx = \frac{3h}{10}\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right)$$

$$\Rightarrow \int_0^1 \frac{x}{1+x^2}\,dx = \frac{3(1/6)}{10}\left[0 + 0.8105 + 0.3 + 2.4 + 0.4615 + 2.4590 + 0.5\right]$$

$$\Rightarrow \int_0^1 \frac{x}{1+x^2}\,dx = 0.3466$$

$$\Rightarrow \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = 0.3466$$

$$\log(1+x^2)\Big|_0^1 = 2\times 0.3466$$

$$\log(2) - \log(1) = 0.6932$$

$$\log_e^2 = 0.6932$$

E7 Use Simpson's $\frac{1}{3}$rd rule 7 ordinates to Evaluate $\int_2^8 \frac{1}{\log_e^x} dx$

Given :- $y = \frac{1}{\log_{10}^x}$

$$a=2,\ b=8,\ n=6$$

$$h = \frac{b-a}{n} = \frac{8-2}{6} = 1$$

$$P = \{a = x_0 = 2,\ x_1 = 3,\ x_2 = 4,\ x_3 = 5,\ x_4 = 6,\ x_5 = 7,\ x_6 = 8 = b\}$$

x	$y = \frac{1}{\log_{10}^x}$
2	$y_0 = 3.3219$
3	$y_1 = 2.0460$
4	$y_2 = 1.6609$
5	$y_3 = 1.4306$
6	$y_4 = 1.2851$
7	$y_5 = 1.1833$
8	$y_6 = 1.1073$

By Simpson's 1/3rd rule

$$n=6$$

$$\Rightarrow \int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_0) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$\Rightarrow \int_2^8 \frac{1}{\log_{10}^x} dx = \frac{1}{3}\Big[(3.3219 + 1.1093) + 2(1.6609 + 1.285) + 4(2.096 + 1.4306 + 1.1833)\Big]$$

$\Rightarrow \frac{1}{3}\left(4.4292 + 5.6920 + 8.7596\right)$

$$\int_2^8 \frac{1}{\log_{10}^x}\,dx = 9.6936$$

1B Use Simpson's $\frac{1}{3}$rd rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 Subintervals

<u>Given</u> :- $y = e^{-x^2}$

$a=0, \quad b=0.6, \quad n=6$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

$p = \{x_0 = 0,\ x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4,\ x_5 = 0.5,\ x_6 = 0.6\}$

x	e^{-x^2}
0	$e^{0.1} = 1 \Rightarrow y_0$
0.1	$e^{-0.01} = 0.99 \Rightarrow y_1$
0.2	$e^{-0.04} = 0.9607 \Rightarrow y_2$
0.3	$e^{-0.09} = 0.9137 \Rightarrow y_3$
0.4	$e^{-0.16} = 0.8521 \Rightarrow y_4$
0.5	$e^{-0.25} = 0.7788 \Rightarrow y_5$
0.6	$e^{-0.36} = 0.697 \Rightarrow y_6$

Numerical Soln for transcidental Equation

An Eqn which involves algebraic logarithmic Exponential Trignometry function is called the transcidental Eqn

Regular falsi method (or) falsi position method

Step 1:- Write the given transcidental Eqn in the form of $f(x)=0$

Step 2:- Choose x_0 and x_1 nearest to the real root for which $f(x_0)<0$, $f(x_0)>0$

Step 3:-

Let the real root $x_2 = \dfrac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)}$

Suppose $f(x_2)<0$. We Say that the root lies b/w x_2 and x_1 and as follows

Step 4:-

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

If $f(x_3)>0$ We Say that the roots lies b/w x_2 and x_3 and follows the Same

Step 5:-

Continue the Same procedure until $f(x_n)$ is 0 (or) approximatly zero

[19] find the real root of the equation $xe^x - 3 = 0$ by regula falsi method correct to 3 decimal places

Soln $xe^x - 3 = 0$

$\Longrightarrow f(x) = 0$

$f(x) = xe^x - 3$

Let $x_0 = 0$, $x_1 = 1.1$

$f(x_0) = f(1) = 1 \cdot e^1 - 3 = -0.2817 < 0$

$f(x_1) = f(1.1) = 1.1 e^{1.1} - 3 = 0.30467$

the roots lies between x_0 and x_1

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{(1)(0.3046) - (1.1)(-0.2817)}{0.3046 + 0.2817}$$

$\Rightarrow x_2 = 1.0480$

$\therefore f(x_2) = (1.048) e^{1.048} - 3 = -0.011 < 0$

the roots lies b/w x_2 and x_1

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(1.048)(0.3046) - (1.1)(-0.011)}{0.3046 + 0.011}$$

$\Rightarrow x_3 = 1.0497$

$\therefore f(x_3) = (1.0497) e^{1.044} - 3 = -0.0012 < 0$

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{(1.0497)(0.3046) - (1.1)(-0.0012)}{0.3046 + 0.0012}$$

$\Rightarrow x_4 = 1.0498$

$f(x_4) = (1.049 + e^{1.0498} - 3) = 0.0006 \approx 0$

the real root is $x = 1.0498$

20. Find the real root of Eqn $xe^x - 2 = 0$ by regular-false method

Soln :-

Given :- $xe^x - 2 = 0$

$$f(x) = 0$$

$$\therefore f(x) = xe^x - 2$$

Let $x_0 = 0.8$, $x_1 = 0.9$

$$\Rightarrow f(x_0) = f(0.8) = -0.2195 < 0$$

$$f(x_1) = f(0.9) = 0.2136 > 0$$

$$\Rightarrow x_2 = \frac{(0.8)(0.2136) - (0.9)(-0.2195)}{0.2136 + 0.2195} = 0.8506$$

$$\therefore f(x_2) = f(0.8536) = -0.0087 < 0$$

$$\Rightarrow x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(0.8506)(0.2136) - (0.9)(-0.0087)}{0.2136 + 0.0087}$$

$$x_3 = 0.08525$$

$$\therefore f(x_3) = f(0.8525) = -0.0004 < 0$$

$$\Rightarrow x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{(0.8525)(0.2136) - (0.9)(-0.0004)}{0.2136 + 0.0004}$$

$$x_4 = 0.08526$$

$$\therefore f(x_4) = f(0.8526) = -0.00002 < 0$$

$\therefore \ f(x_4) = f(0.8526) = -0.00002 < 0$

$$\boxed{\text{The real root of the Eqn is } x = 0.8526}$$

21) Use the regula falsi method to obtain a root of the Eqn $2x - \log_{10} x = 7$ which lies b/w 3.5 and 4 carry out 2 itteration

Soln:- $2x - \log_{10} x = 7$

$\Rightarrow 2x - \log_{10} x - 7 = 0$

$f(x) = 0$

$f(x) = 2x - \log_{10} x - 7$

and $x_0 = 3.5$ and $x_1 = 4$

$f(x_0) = f(3.5) = 2(3.5) - \log_{10}(3.5) - 7 = -0.5441 < 0$

$f(x_1) = f(4) = 2(4) - \log_{10}(4) - 7 = 0.397 > 0$

the roots are lies b/w x_0 and x_1

$$x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)}$$

$$x_2 = \frac{4(-0.544) - (3.5)(0.3970)}{-0.5441 - 0.397}$$

$x_2 = 3.788$

$f(x_2) = 2(3.7882) - \log_{10}(3.788) - 7 = 0.0009 < 0$

the roots lies b/w x_2 and x_1

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(3.788)(0.3979) - (4)(-0.0009)}{0.3978 + 0.0009}$$

$x_3 = 3.7892$

$f(x_3) = 2(3.7892) - \log_{10}(3.7892) - 7$

$f(x_3) = -0.0001 \approx 0$

$$\boxed{\text{The roots lies b/w } x = 3.7892}$$

22) Find the real roots of $x\log_{10}x - 1.2 = 0$ correct to 3 decimal places lying in the interval (2,3) using regula falsi method

Soln :- Given :- $x\log_{10}x - 1.2 = 0$

$f(x) = 0$

$f(x) = x\log_{10}x - 1.2$

Let $x_0 = 2.7$, $x_1 = 2.8$

$\Rightarrow f(x_0) = f(2.7) = -0.0353 < 0$

$f(x_1) = f(2.8) = 0.0520 > 0$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(2.8)(0.0520) - (2.9)(-0.0353)}{0.0520 + 0.0353} = 2.7404$$

$f(x_2) = f(2.7404) = -0.0002 < 0$

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(2.7404)(0.05201) - (2.9)(-0.0002)}{0.0520 + 0.0002}$$

$x_3 = 2.7406$

$f(x_3) = f(2.7406) = -0.00002 < 0$

$$\boxed{\text{The real root of the Eqn } x = 2.7406}$$

Q3 Find the 4^{th} root of 12 by using regula - falsi method

Soln :- Let $x = \sqrt[4]{12}$

$\Rightarrow x^4 = 12$

$\Rightarrow x^4 - 12 = 0$

$\Rightarrow f(x) = 0$

$f(x) = x^4 - 12$

Let $x_0 = 1.8$, $x_1 = 1.9$

$\therefore f(x_0) = -1.5024 < 0$

$f(x_1) = 1.0321 > 0$

$\therefore$ the root lies between x_0 and x_1

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(1.8)(1.0321) - (1.9)(-1.5024)}{1.0321 + 1.5024}$$

$\Rightarrow x_2 = 1.8592$

$\therefore f(x_2) = -0.0517 < 0$

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(1.8592)(1.0321) - (1.9)(-0.0517)}{1.0321 + 0.0517}$$

$\Rightarrow x_3 = 1.8611$

$\therefore f(x_3) = 0.0028 < 0$

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$= \frac{(1.8611)(1.0321) - (1.9)(-0.0028)}{1.0321 + 0.0028}$$

$x_4 = 1.8612$

$f(x_4) = -0.0001 \approx 0$

$x = 1.8612$ is real root

$\therefore \sqrt[4]{12} = 1.8612$

24 Using regula falsi method to find the real root for the Equation $x^3 - 2x - 5 = 0$

Soln :-

$x^3 - 2x - 5 = 0$

$\Rightarrow f(x) = 0$

$f(x) = x^3 - 2x - 5 = 0$

Let $x_0 = 2 \Rightarrow f(x_0) = -1 < 0$

$x_1 = 2.1 \Rightarrow f(x_1) = 0.061 > 0$

$$\therefore x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(2)(0.06) - (2.1)(-1)}{0.061 + 1}$$

$\Rightarrow x_2 = 2.0947$

$f(x_2) = -0.0039 < 0$

$$\therefore x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(2.0942)(0.061) - (2.1)(-0.0039)}{0.061 + 0.0039}$$

$\Rightarrow x_3 = 2.0947$

$f(x_3) = -0.0005 < 0$

$$\therefore x_4 = \frac{(2.0945)(0.061) - (2.1)(-0.0005)}{0.061 + 0.0005}$$

$\Rightarrow x_4 = 2.0945$

$f(x_4) = -0.0005 \approx 0$

$\therefore x = 2.0945$ is real root

Newton Raphson Method

Step 1 :- Write the given transcidental Eqn in the form of $f(x)=0$ and find $f'(x)$

Step 2 :- Choose x_0. for which $f(x_0) < 0$

Step 3 :- Use of formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 0,1,2$ ----

for the real root and continue the Same until to get the value of x has to be Equal

25 Find $\sqrt[3]{37}$ by newton Raphson method

Soln :-

Let $x = \sqrt[3]{37}$

$\Rightarrow x^3 = 37$

$\Rightarrow x^3 - 37 = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = x^3 - 37 \Rightarrow f'(x) = 3x^2$

Let $x_0 = 3.3$

$\Rightarrow f(x_0) = -1.063, \quad f'(x_0) = 32.6700$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\Rightarrow 3.3 - \frac{(-1.063)}{32.67}$

$\Rightarrow 3.3325$

$\Rightarrow f(x_1) = -0.0092, \quad f'(x_1) = 33.3166$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\Rightarrow 3.3325 - \frac{(-0.0092)}{33.3166}$

$\Rightarrow 3.3327$

$f(x_2) = 0.0159 \qquad f'(x_2) = 33.3206$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$\Rightarrow 3.3327 - \frac{(0.0159)}{33.3206}$

$\Rightarrow 3.3322$

$x = \sqrt[3]{27}$

$x = 3.3322$

26 find a real root of eqn $x\sin x + \cos x = 0$ near to $x = \pi$ radians correct to 4 decimal places by newtons Raphson method

Soln :-

$x\sin x + \cos x = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = x\sin x + \cos x$

$\Rightarrow f'(x) = x\cos x + \sin x - \sin x$

$f'(x) = x\cos x$

and $x_0 = \pi$

$\Rightarrow f(x_0) = \pi\sin\pi + \cos\pi = -1$

$f'(x_0) = \pi\cos\pi = -\pi$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\Rightarrow \pi - \frac{(-1)}{(-\pi)}$

$\Rightarrow \pi - \frac{1}{\pi}$

$\Rightarrow \frac{22}{7} - \frac{7}{22}$

$\Rightarrow x_1 \Rightarrow 2.8246$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow 2.8246 - \frac{(-0.0697)}{(-2.6838)} = 2.7986$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow 2.7986 - \frac{(-0.00056)}{(-2.6355)} = 2.7983$

$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \Rightarrow 2.7983 - \frac{(0.00022)}{(-2.6350)} = 2.7983$

$x = 2.7983$ is a real root

27 Using newton raphson method find the real root of Equation $3x = \cos x + 1$ [Use radian mode]

Soln :- given :- $3x = \cos x + 1$

$\Rightarrow 3x - \cos x - 1 = 0$

$\Rightarrow f(x) = 0$

$f(x) = 3x - \cos x - 1$

$\Rightarrow f'(x) = 3 + \sin x$

Let $x_0 = 0.5$

$\Rightarrow f(x_0) = 0.3715 < 0$

$f'(x_0) = 3.4794$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(-0.3735)}{3.4794} = 0.6084$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6084 - \frac{0.00463}{3.5715} = 0.6071$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071 - \frac{(-0.0000058)}{3.5704} = 0.6071$

$x = 0.6071$ is real root

28 Find the real root of $x\log_{10}^{x} = 1.2$ using newton raphson method near to 2.5

Soln :- Given :- $x\log_{10}^{x} = 1.2$

$$\Rightarrow x\log_{10}^{x} - 1.2 = 0$$

$$\Rightarrow \frac{x\log_e^x}{\log_e 10} - 1.2 = 0$$

$$\Rightarrow (0.4343)\, x\log x - 1.2 = 0$$

$$\Rightarrow f(x) = 0$$

$$\therefore f(x) = (0.4343)x\log x - 1.2$$

$$f'(x) = (0.4343)\left[x \cdot \frac{1}{x} + (1)\log x\right] - 0$$

$$\Rightarrow f'(x) = (0.4343)(1+\log x)$$

and $x_0 = 2.5$

$$f(x_0) = -0.2051$$

$$f'(x_0) = 0.8322$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow 2.5 - \frac{(-0.2051)}{(0.8322)} = 2.7465$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow 2.7465 - \frac{(0.0051)}{(0.8730)} = 2.7406$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow 2.7406 - \frac{(0.00002)}{(0.8722)} = 2.7406$$

$\boxed{\text{The real root is } 2.7406}$

29 Using newton raphson method find the real root that lies near $x = 4.5$ of the Equation $\tan x = x$ correct to 4 decimal places [take x as a radian]

Soln Given, $\tan x = x$

$$f(x) = x - \tan x$$

$$f'(x) = 1 - \sec^2 x$$

$f'(x) = -(\sec^2 x - 1)$

$f'(x) = -\tan^2 x$

and $x_0 = 4.5$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow 4.5 - \frac{(-0.1373)}{(-21.5048)} = 4.4936$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow 4.4936 - \frac{(-0.0039)}{(-20.227)} = 4.4934$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow 4.493 - \frac{(0.0002)}{(-20.1869)} = 4.4934$

$\boxed{\text{The real root is } x = 4.4934}$

30] find the real root of the Eqn $xe^x - \cos x = 0$ Correct to 3 decimal places by using newton drappon method near To the root $x = 0.5$

Soln :- $xe^x - \cos x = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = xe^x - \cos x$

$f'(x) = 1.e^x + xe^x + \sin x$

$\Rightarrow f'(x) = e^x(1+x) + \sin x$

and $x_0 = 0.5$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow 0.5 - \frac{(-0.0532)}{2.9525} = 0.5180$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow 0.5180 - \frac{(0.0007)}{(3.0434)} = 0.5177$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \Rightarrow 0.5177 - \frac{(-0.0001)}{(3.0418)} = 0.5177$

$\boxed{\text{The real root is } x = 0.5177}$